1. Use the Euclidean Algorithm to find the greatest common divisor of each of the following pairs of numbers.

   (a) \( a = 105, b = 56 \)
   (b) \( a = 162, b = 47 \)
   (c) \( a = -2485, b = 1704 \)
   (d) \( a = 924, b = 1960 \)
   (e) \( a = -8120, b = -14355 \)

2. Let \( a \) and \( b \) be integers, not both zero, and let \( g = \text{gcd}(a, b) \). Prove that any common divisor of \( a \) and \( b \) must also divide \( g \). (Hint: Use Bezout’s Identity.)

3. Let \( a \) and \( b \) be integers, not both zero, and let \( g = \text{gcd}(a, b) \). Since \( g \mid a \), there exists an integer \( a' \) such that \( a = ga' \), and likewise there is an integer \( b' \) such that \( b = gb' \). (This is a fancy way of saying “Let \( a' = \frac{a}{g} \) and \( b' = \frac{b}{g} \), but without using fractions!”) Prove that \( a' \) and \( b' \) are relatively prime, i.e. that \( \text{gcd}(a', b') = 1 \).

4. For each of the following pairs of numbers \( a \) and \( b \), use the Euclidean Algorithm to find their greatest common divisor \( g \). Then use that to find integers \( x \) and \( y \) such that \( ax + by = g \), as guaranteed by Bezout’s Identity. (That is, apply the “Extended Euclidean Algorithm”.)

   (a) \( a = 509, b = 94 \)
   (b) \( a = -1260, b = 816 \)

The integers \( x \) and \( y \) given by Bezout’s Identity (and which you can calculate using the Extended Euclidean Algorithm, as in the previous problem) are not unique. The next four problems will explore the extent of that non-uniqueness.

5. Let \( a \) and \( b \) be relatively prime integers (so that \( \text{gcd}(a, b) = 1 \)). Suppose you are given \( x_0, y_0 \in \mathbb{Z} \) such that \( ax_0 + by_0 = 1 \). Define, for any \( k \in \mathbb{Z} \),

   \[ x_k = x_0 + bk \quad \text{and} \quad y_k = y_0 - ak. \]

   (a) Show that for all \( k \in \mathbb{Z} \), \( ax_k + by_k = 1 \).
   
   (b) Now suppose that \( x, y \in \mathbb{Z} \) satisfy \( ax + by = 1 \). Prove that there is some \( k \in \mathbb{Z} \) for which \( x = x_k \) and \( y = y_k \). (Hint: You might need to use the Generalized Euclid’s Lemma, which should be stated in class this week. Or you might not.)
6. Now let $a$ and $b$ be any integers, not both zero, and let $g = \gcd(a, b)$. Suppose you are given $x_0, y_0 \in \mathbb{Z}$ such that $ax_0 + by_0 = g$. Let $a'$ and $b'$ be as in problem 3, so that $\gcd(a', b') = 1$. Define, for any $k \in \mathbb{Z}$,

$$x_k = x_0 + b'k \quad \text{and} \quad y_k = y_0 - a'k.$$ 

*Note: Try to avoid the use of fractions throughout this problem! You don’t actually need them anywhere!*

(a) Show that for all $k \in \mathbb{Z}$, $ax_k + by_k = g$.

(b) Now suppose that $x, y \in \mathbb{Z}$ satisfy $ax + by = g$. Prove that there is some $k \in \mathbb{Z}$ for which $x = x_k$ and $y = y_k$. (*Hint: Just divide by $g$ and you can immediately apply the previous problem.*)

7. Write the set of all solutions $(x, y)$ to the equation $509x + 94y = 1$, using set notation. That is, based on the previous two problems, your answer should be in the form

$$\{ (x + k, y + k) \mid k \in \mathbb{Z} \}$$

Note that you don’t need to do any computations for this, as you can use your answer to problem 4(a).

8. Let $g = \gcd(-1260, 816)$, which you computed in problem 4(b). (So again, no new computations are needed for this problem). Just as you did in the previous problem, use set notation to write the set of all solutions $(x, y)$ to the equation

$$-1260x + 816y = g.$$ 

9. *Extra challenge problem:* We stated the well-ordering principle for the natural numbers in class, as follows: every non-empty set of natural numbers contains a smallest element. In this problem, you will prove this, by using strong mathematical induction.

Let $S$ be a set of natural numbers, and suppose that $S$ does not contain a smallest element. Use a strong induction proof to show that

$$\forall n \in \mathbb{N}, \ n \notin S.$$ 

That is, you’ve shown that $S$ is empty.

The above proves the following statement: for any set $S$ of natural numbers, if $S$ does not contain a smallest element, then $S$ is empty. How does this prove the well-ordering principle for $\mathbb{N}$?