Exercise 1. Let $X$ be a RV taking values from the interval $[a, b]$.

(i) Use the usual ‘completing squares’ trick for a second moment to show that
\[ 0 \leq \mathbb{E}[(X - t)^2] = (t - \mathbb{E}[X])^2 + \text{Var}(X) \quad \forall t \in \mathbb{R}. \quad (1) \]

(ii) Conclude that $\mathbb{E}[(X - t)^2]$ is minimized when $t = \mathbb{E}[X]$ and the minimum is $\text{Var}(X)$.

(iii) By plugging in $t = (a + b)/2$ in (1), show that
\[ \text{Var}(X) = \mathbb{E}[(X - a)(X - b)] + \frac{(b - a)^2}{4} - \left( \mathbb{E}[X] - \frac{a + b}{2} \right)^2. \quad (2) \]

(iv) Show that $\mathbb{E}[(X - a)(X - b)] \leq 0$.

(v) Conclude that $\text{Var}(X) \leq (b - a)^2/4$, where the equality holds if and only if $X$ takes the extreme values $a$ and $b$ with equal probabilities.

Exercise 2 (Monte Carlo integration). Let $(X_k)_{k \geq 1}$ be i.i.d. Uniform([0, 1]) RVs and let $f : [0, 1] \to \mathbb{R}$ be a continuous function. For each $n \geq 1$, let
\[ I_n = \frac{1}{n} \left( f(X_1) + f(X_2) + \cdots + f(X_n) \right). \quad (3) \]

(i) Suppose $f_0^1 |f(x)| \, dx < \infty$. Show that $I_n \to I := \int_0^1 f(x) \, dx$ in probability.

(ii) Further assume that $f_0^1 |f(x)|^2 \, dx < \infty$. Use Chebyshev’s inequality to show that
\[ \mathbb{P}(|I_n - I| \geq a/\sqrt{n}) \leq \frac{\text{Var}(f(X_1))}{a^2} = \frac{1}{a^2} \left( \int_0^1 f(x)^2 \, dx - I^2 \right). \quad (4) \]

Exercise 3. In this exercise, we estimate some partial sums using integral comparison.

(i) For any integer $d \geq 1$, show that
\[ \sum_{k=2}^{n} \frac{1}{k^d} \leq \int_1^n \frac{1}{x^d} \, dx \leq \sum_{k=1}^{n-1} \frac{1}{k^d} \quad (5) \]
by considering the upper and lower sum for the Riemann integral $\int_1^n x^{-d} \, dx$.

(ii) Show that
\[ \log n \leq \sum_{k=1}^{n-1} \frac{1}{k} \leq 1 + \log(n - 1). \quad (6) \]

(iii) Show that for all $d \geq 2$,
\[ \sum_{k=1}^{n-1} \frac{1}{k^d} \leq \sum_{k=1}^{\infty} \frac{1}{k^d} \leq 1 + \int_1^\infty \frac{1}{x^d} \, dx \leq 2. \quad (7) \]

Exercise 4. For each $n \geq 1$, let $X_{1,n}, X_{2,n}, \cdots, X_{n,n}$ be a sequence of independent geometric RVs where $X_{k,n} \sim \text{Geom}(n-k)/n$. Define $\tau^n = X_{1,n} + X_{2,n} + \cdots + X_{n,n}$.

(i) Show that $\mathbb{E}[\tau^n] = n \sum_{k=1}^{n-1} k^{-1}$. Using Exercise 3 (ii), deduce that
\[ n \log n \leq \mathbb{E}[\tau^n] \leq n \log(n - 1) + n. \quad (8) \]

(ii) Using $\text{Var} \left( \text{Geom}(p) \right) = (1 - p)/p^2 \leq p^{-2}$ and Exercise 3 (iii), show that
\[ \text{Var}(\tau^n) \leq n^2 \sum_{k=1}^{n-1} k^{-2} \leq 2n^2. \quad (9) \]
(iii) By Chebyshev’s inequality, show that for each $\varepsilon > 0$,
\[
\mathbb{P}\left( |\tau^n - E[\tau^n]| > \varepsilon n \log n \right) \leq \frac{\text{Var}(\tau^n)}{\varepsilon^2 n^2 \log^2 n} \leq \frac{2}{\varepsilon^2 \log^2 n}.
\] (10)

Conclude that
\[
\frac{\tau^n - E[\tau^n]}{n \log n} \to 0 \quad \text{as } n \to \infty \text{ in probability.}
\] (11)

(iv) By using part (i), conclude that
\[
\frac{\tau^n}{n \log n} \to 1 \quad \text{as } n \to \infty \text{ in probability.}
\] (12)