LIST OF ABBREVIATIONS.

Bs.  
Begriffsschrift. Eine der arithmetischen nachgebildete Formelsprache des reinen  
Denkens. Halle a/S, 1879.

Gl.  
Grundlagen der Arithmetik. Eine logisch-mathematische Untersuchung über den  
Begriff der Zahl. Breslau, 1884.

FT.  
Ueber formale Theorien der Arithmetik. Sitzungsberichte der Jenaischen Gesells-  
schaft für Medicin und Naturwissenschaft, 1885.

FuB.  

BuG.  

SuB.  

KB.  
Kritische Beleuchtung einiger Punkte in E. Schröder's Vorlesungen über die Algebra  

BP.  
Ueber die Begriffsschrift des Herrn Peano und meine eigene. Berichte der math.-  
physischen Classe der Königl. Sächs. Gesellschaft der Wissenschaften zu Leipzig  
(1896).

Gg.  
Vol. ii. 1903.
APPENDIX A.

THE LOGICAL AND ARITHMETICAL DOCTRINES OF FREGE.

475. The work of Frege, which appears to be far less known than it
deserves, contains many of the doctrines set forth in Parts I and II of the
present work, and where it differs from the views which I have advocated,
the differences demand discussion. Frege's work abounds in subtle distinc-
tions, and avoids all the usual fallacies which beset writers on Logic. His
symbolism, though unfortunately so cumbersome as to be very difficult to
employ in practice, is based upon an analysis of logical notions much more
profound than Peano's, and is philosophically very superior to its more
convenient rival. In what follows, I shall try briefly to expound Frege's
theories on the most important points, and to explain my grounds for
differing where I do differ. But the points of disagreement are very few
and slight compared to those of agreement. They all result from difference
on three points: (1) Frege does not think that there is a contradiction in the
notion of concepts which cannot be made logical subjects (see § 40 supra);
(2) he thinks that, if a term a occurs in a proposition, the proposition can
always be analysed into a and an assertion about a (see Chapter vii);
(3) he is not aware of the contradiction discussed in Chapter x. These are
very fundamental matters, and it will be well here to discuss them afresh,
since the previous discussion was written in almost complete ignorance of
Frege's work.

Frege is compelled, as I have been, to employ common words in technical
senses which depart more or less from usage. As his departures are frequently
different from mine, a difficulty arises as regards the translation of his terms.
Some of these, to avoid confusion, I shall leave untranslated, since every
English equivalent that I can think of has been already employed by me in a
slightly different sense.

The principal heads under which Frege's doctrines may be discussed are
the following: (1) meaning and indication; (2) truth-values and judgment;
(3) Begriff and Gegenstand; (4) classes; (5) implication and symbolic logic;
(6) the definition of integers and the principle of abstraction; (7) mathema-
tical induction and the theory of progressions. I shall deal successively
with these topics.
476. Meaning and indication. The distinction between meaning (Sinn) and indication (Bedeutung)* is roughly, though not exactly, equivalent to my distinction between a concept as such and what the concept denotes (§ 96). Frege did not possess this distinction in the first two of the works under consideration (the Begriffsschrift and the Grundlagen der Arithmetik); it appears first in BuG. (cf. p. 198), and is specially dealt with in SuB. Before making the distinction, he thought that identity has to do with the names of objects (Bs. p. 13): “A is identical with B” means, he says, that the sign A and the sign B have the same signification (Bs. p. 15)—a definition which, verbally at least, suffers from circularity. But later he explains identity in much the same way as it was explained in § 64. “Identity,” he says, “calls for reflection owing to questions which attach to it and are not quite easy to answer. Is it a relation†? A relation between Gegenstände? or between names or signs of Gegenstände?” (SuB. p. 25). We must distinguish, he says, the meaning, in which is contained the way of being given, from what is indicated (from the Bedeutung). Thus “the evening star” and “the morning star” have the same indication, but not the same meaning. A word ordinarily stands for its indication; if we wish to speak of its meaning, we must use inverted commas or some such device (pp. 27–8). The indication of a proper name is the object which it indicates; the presentation which goes with it is quite subjective; between the two lies the meaning, which is not subjective and yet is not the object (p. 30). A proper name expresses its meaning, and indicates its indication (p. 31).

This theory of indication is more sweeping and general than mine, as appears from the fact that every proper name is supposed to have the two sides. It seems to me that only such proper names as are derived from concepts by means of the can be said to have meaning, and that such words as John merely indicate without meaning. If one allows, as I do, that concepts can be objects and have proper names, it seems fairly evident that their proper names, as a rule, will indicate them without having any distinct meaning; but the opposite view, though it leads to an endless regress, does not appear to be logically impossible. The further discussion of this point must be postponed until we come to Frege’s theory of Begriffe.

477. Truth-values and Judgment. The problem to be discussed under this head is the same as the one raised in § 52†, concerning the difference between asserted and unasserted propositions. But Frege’s position on this question is more subtle than mine, and involves a more radical analysis of judgment. His Begriffsschrift, owing to the absence of the distinction between meaning and indication, has a simpler theory than his later works. I shall therefore omit it from the discussions.

There are, we are told (Gg. p. x), three elements in judgment: (1) the recognition of truth, (2) the Gedanke, (3) the truth-value (Wahrheitswürth).

* I do not translate Bedeutung by denotation, because this word has a technical meaning different from Frege’s, and also because bedeuten, for him, is not quite the same as denoting for me.

† This is the logical side of the problem of Annahmen, raised by Meinong in his able work on the subject, Leipzig, 1902. The logical, though not the psychological, part of Meinong’s work appears to have been completely anticipated by Frege.
Here the Gedanke is what I have called an unasserted proposition—or rather, what I called by this name covers both the Gedanke alone and the Gedanke together with its truth-value. It will be well to have names for these two distinct notions; I shall call the Gedanke alone a *propositional concept*; the truth-value of a Gedanke I shall call an *assumption*. Formally at least, an assumption does not require that its content should be a propositional concept: whatever \( x \) may be, "the truth of \( x \)" is a definite notion. This means the true if \( x \) is true, and if \( x \) is false or not a proposition it means the false (FuB. p. 21). In like manner, according to Frege, there is "the falsehood of \( x \)"; these are not assertions and negations of propositions, but only assertions of truth or of falsity, i.e., negation belongs to what is asserted, and is not the opposite of assertion†. Thus we have first a propositional concept, next its truth or falsity as the case may be, and finally the assertion of its truth or falsity. Thus in a hypothetical judgment, we have a relation, not of two judgments, but of two propositional concepts (SuB. p. 43).

This theory is connected in a very curious way with the theory of meaning and indication. It is held that every assumption indicates the true or the false (which are called truth-values), while it means the corresponding propositional concept. The assumption "2 = 4" indicates the true, we are told, just as "2" indicates 4‡ (FuB. p. 13; SuB. p. 32). In a dependent clause, or where a name occurs (such as Odysseus) which indicates nothing, a sentence may have no indication. But when a sentence has a truth-value, this is its indication. Thus every assertive sentence (Behauptungssatz) is a proper name, which indicates the true or the false (SuB. pp. 32—4; Gg. p. 7). The sign of judgment (Urtheilstrich) does not combine with other signs to denote an object; a judgment indicates nothing, but asserts something. Frege has a special symbol for judgment, which is something distinct from and additional to the truth-value of a propositional concept (Gg. pp. 9—10).

478. There are some difficulties in the above theory which it will be well to discuss. In the first place, it seems doubtful whether the introduction of truth-values marks any real analysis. If we consider, say, "Caesar died," it would seem that what is asserted is the propositional concept "the death of Caesar," not "the truth of the death of Caesar." This latter seems to be merely another propositional concept, asserted in "the death of Caesar is true," which is not, I think, the same proposition as "Caesar died." There is great difficulty in avoiding psychological elements here, and it would seem that Frege has allowed them to intrude in describing judgment as the recognition of truth (Gg. p. x). The difficulty is due to the fact that there is a psychological sense of assertion, which is what is lacking to Meinong's *Annahmen*, and that this does not run parallel with the logical sense. Psychologically, any proposition, whether true or false, may be merely thought of, or may be actually asserted: but for this possibility, error would be impossible. But logically, true propositions only are asserted,

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* Frege, like Meinong, calls this an *Annahme* : FuB. p. 21.
† Gg. p. 10. Cf. also Bs. p. 4.
‡ When a term which indicates is itself to be spoken of, as opposed to what it indicates, Frege uses inverted commas. Cf. § 56.
though they may occur in an unasserted form as parts of other propositions. In "p implies q," either or both of the propositions p, q may be true, yet each, in this proposition, is unasserted in a logical, and not merely in a psychological, sense. Thus assertion has a definite place among logical notions, though there is a psychological notion of assertion to which nothing logical corresponds. But assertion does not seem to be a constituent of an asserted proposition, although it is, in some sense, contained in an asserted proposition. If p is a proposition, "p's truth" is a concept which has being even if p is false, and thus "p's truth" is not the same as p asserted. Thus no concept can be found which is equivalent to p asserted, and therefore assertion is not a constituent in p asserted. Yet assertion is not a term to which p, when asserted, has an external relation; for any such relation would need to be itself asserted in order to yield what we want. Also a difficulty arises owing to the apparent fact, which may however be doubted, that an asserted proposition can never be part of another proposition: thus, if this be a fact, where any statement is made about p asserted, it is not really about p asserted, but only about the assertion of p. This difficulty becomes serious in the case of Frege's one and only principle of inference (Bs. p. 9): "p is true and p implies q; therefore q is true." Here it is quite essential that there should be three actual assertions, otherwise the assertion of propositions deduced from asserted premises would be impossible; yet the three assertions together form one proposition, whose unity is shown by the word therefore, without which q would not have been deduced, but would have been asserted as a fresh premise.

It is also almost impossible, at least to me, to divorce assertion from truth, as Frege does. An asserted proposition, it would seem, must be the same as a true proposition. We may allow that negation belongs to the content of a proposition (Bs. p. 4), and regard every assertion as asserting something to be true. We shall then correlate p and not-p as unasserted propositions, and regard "p is false" as meaning "not-p is true." But to divorce assertion from truth seems only possible by taking assertion in a psychological sense.

479. Frege's theory that assumptions are proper names for the true or the false, as the case may be, appears to me also untenable. Direct inspection seems to show that the relation of a proposition to the true or the false is quite different from that of (say), "the present King of England" to Edward VII. Moreover, if Frege's view were correct on this point, we should have to hold that in an asserted proposition it is the meaning, not the indication, that is asserted, for otherwise, all asserted propositions would assert the very same thing, namely the true, (for false propositions are not asserted). Thus asserted propositions would not differ from one another in any way, but would be all strictly and simply identical. Asserted propositions have no indication (FuB. p. 21), and can only differ, if at all, in some way analogous to meaning. Thus the meaning of the unasserted proposition together with its truth-value must be what is asserted,

* Cf. supra, § 18, (4) and § 38.
if the meaning simply is rejected. But there seems no purpose in introducing the truth-value here: it seems quite sufficient to say that an asserted proposition is one whose meaning is true, and that to say the meaning is true is the same as to say the meaning is asserted. We might then conclude that true propositions, even when they occur as parts of others, are always and essentially asserted, while false propositions are always unasserted, thus escaping the difficulty about therefore discussed above. It may also be objected to Frege that "the true" and "the false," as opposed to truth and falsehood, do not denote single definite things, but rather the classes of true and false propositions respectively. This objection, however, would be met by his theory of ranges, which correspond approximately to my classes; these, he says, are things, and the true and the false are ranges (v. infra).

480. Begriff and Gegenstand. Functions. I come now to a point in which Frege's work is very important, and requires careful examination. His use of the word Begriff does not correspond exactly to any notion in my vocabulary, though it comes very near to the notion of an assertion as defined in § 43, and discussed in Chapter VII. On the other hand, his Gegenstand seems to correspond exactly to what I have called a thing (§ 48). I shall therefore translate Gegenstand by thing. The meaning of proper name seems to be the same for him as for me, but he regards the range of proper names as confined to things, because they alone, in his opinion, can be logical subjects.

Frege's theory of functions and Begriffe is set forth simply in FuB. and defended against the criticisms of Kerry* in BuG. He regards functions—and in this I agree with him—as more fundamental than predicates and relations; but he adopts concerning functions the theory of subject and assertion which we discussed and rejected in Chapter VII. The acceptance of this view gives a simplicity to his exposition which I have been unable to attain; but I do not find anything in his work to persuade me of the legitimacy of his analysis.

An arithmetical function, e.g. \(2x^2 + x\), does not denote, Frege says, the result of an arithmetical operation, for that is merely a number, which would be nothing new (FuB. p. 5). The essence of a function is what is left when the \(x\) is taken away, i.e., in the above instance, \(2(\ )^2 + (\ )\). The argument \(x\) does not belong to the function, but the two together make a whole (ib. p. 6). A function may be a proposition for every value of the variable; its value is then always a truth-value (p. 13). A proposition may be divided into two parts, as "Caesar" and "conquered Gaul." The former Frege calls the argument, the latter the function. Any thing whatever is a possible argument for a function (p. 17). (This division of propositions corresponds exactly to my subject and assertion as explained in § 43, but Frege does not restrict this method of analysis as I do in Chapter VII.) A thing is anything which is not a function, i.e. whose expression leaves no empty place. The two following accounts of the nature of a function are quoted from the earliest and one of the latest of Frege's works respectively.

(1) "If in an expression, whose content need not be propositional

(beurtheilbar), a simple or composite sign occurs in one or more places, and we regard it as replaceable, in one or more of these places, by something else, but by the same everywhere, then we call the part of the expression which remains invariant in this process a function, and the replaceable part we call its argument” (Bs. p. 16).

(2) “If from a proper name we exclude a proper name, which is part or the whole of the first, in some or all of the places where it occurs, but in such a way that these places remain recognizable as to be filled by one and the same arbitrary proper name (as argument positions of the first kind), I call what we thereby obtain the name of a function of the first order with one argument. Such a name, together with a proper name which fills the argument-places, forms a proper name” (Gg. p. 44).

The latter definition may become plainer by the help of some examples. “The present king of England” is, according to Frege, a proper name, and “England” is a proper name which is part of it. Thus here we may regard England as the argument, and “the present king of” as function. Thus we are led to “the present king of x.” This expression will always have a meaning, but it will not have an indication except for those values of x which at present are monarchies. The above function is not propositional. But “Caesar conquered Gaul” leads to “x conquered Gaul”; here we have a propositional function. There is here a minor point to be noticed: the asserted proposition is not a proper name, but only the assumption is a proper name for the true or the false (v. supra); thus it is not “Caesar conquered Gaul” as asserted, but only the corresponding assumption, that is involved in the genesis of a propositional function. This is indeed sufficiently obvious, since we wish x to be able to be any thing in “x conquered Gaul,” whereas there is no such asserted proposition except when x did actually perform this feat. Again consider “Socrates is a man implies Socrates is a mortal.” This (unasserted) is, according to Frege, a proper name for the true. By varying the proper name “Socrates,” we can obtain three propositional functions, namely “x is a man implies Socrates is a mortal,” “Socrates is a man implies x is a mortal,” “x is a man implies x is a mortal.” Of these the first and third are true for all values of x, the second is true when and only when x is a mortal.

By suppressing in like manner a proper name in the name of a function of the first order with one argument, we obtain the name of a function of the first order with two arguments (Gg. p. 44). Thus e.g. starting from “1 < 2,” we get first “x < 2,” which is the name of a function of the first order with one argument, and thence “x < y,” which is the name of a function of the first order with two arguments. By suppressing a function in like manner, Frege says, we obtain the name of a function of the second order (Gg. p. 44). Thus e.g. the assertion of existence in the mathematical sense is a function of the second order: “There is at least one value of x satisfying φx” is not a function of x, but may be regarded as a function of φ. Here φ must on no account be a thing, but may be any function. Thus this proposition, considered as a function of φ, is quite different from functions of the first order, by the fact that the possible arguments are different. Thus given any proposition, say f(a), we may consider either f(x), the function of the first
order resulting from varying $\alpha$ and keeping $f$ constant, or $\phi(\alpha)$, the function of the second order got by varying $f$ and keeping $\alpha$ fixed; or, finally, we may consider $\phi(x)$, in which both $f$ and $\alpha$ are separately varied. (It is to be observed that such notions as $\phi(\alpha)$, in which we consider any proposition concerning $\alpha$, are involved in the identity of indiscernibles as stated in § 43.) Functions of the first order with two variables, Frege points out, express relations (Bs. p. 17); the referent and the relatum are both subjects in a relational proposition (Gl. p. 82). Relations, just as much as predicates, belong, Frege rightly says, to pure logic (ib. p. 83).

481. The word Begriff is used by Frege to mean nearly the same thing as *propositional function* (e.g. FuB. p. 28)*; when there are two variables, the Begriff is a relation. A thing is anything not a function, i.e. anything whose expression leaves no empty place (ib. p. 18). To Frege's theory of the essential cleavage between things and Begriffe, Kerry objects (loc. cit. p. 272 ff.) that Begriffe also can occur as subjects. To this Frege makes two replies. In the first place, it is, he says, an important distinction that some terms can only occur as subjects, while others can occur also as concepts, even if Begriffe can also occur as subjects (BuG. p. 195). In this I agree with him entirely; the distinction is the one employed in §§ 48, 49. But he goes on to a second point which appears to me mistaken. We can, he says, have a concept falling under a higher one (as Socrates falls under man, he means, not as Greek falls under man); but in such cases, it is not the concept itself, but its name, that is in question (BuG. p. 195). “The concept horse,” he says, is not a concept, but a thing; the peculiar use is indicated by inverted commas (ib. p. 196). But a few pages later he makes statements which seem to involve a different view. A concept, he says, is essentially predicative even when something is asserted of it; an assertion which can be made of a concept does not fit an object. When a thing is said to fall under a concept, and when a concept is said to fall under a higher concept, the two relations involved, though similar, are not the same (ib. p. 201). It is difficult to me to reconcile these remarks with those of p. 195; but I shall return to this point shortly.

Frege recognizes the unity of a proposition: of the parts of a propositional concept, he says, not all can be complete, but one at least must be incomplete (*ungesättigt*) or predicative, otherwise the parts would not cohere (ib. p. 205). He recognizes also, though he does not discuss, the oddities resulting from *any* and *every* and such words: thus he remarks that every positive integer is the sum of four squares, but “every positive integer” is not a possible value of $x$ in “$x$ is the sum of four squares.” The meaning of “every positive integer,” he says, depends upon the context (Bs. p. 17)—a remark which is doubtless correct, but does not exhaust the subject. *Self-contradictory* notions are admitted as concepts: $F$ is a concept if “$a$ falls under the concept $F$” is a proposition whatever thing $a$ may be (Gl. p. 87). A concept is the indication of a predicate; a thing is what can never be

* "We have here a function whose value is always a truth-value. Such functions with one argument we have called Begriffe; with two, we call them relations." Cf. Gl. pp. 82—3.
the whole indication of a predicate, though it may be that of a subject (BuG. p. 198).

482. The above theory, in spite of close resemblance, differs in some important points from the theory set forth in Part I above. Before examining the differences, I shall briefly recapitulate my own theory.

Given any propositional concept, or any unity (see § 136), which may in the limit be simple, its constituents are in general of two sorts: (1) those which may be replaced by anything else whatever without destroying the unity of the whole; (2) those which have not this property. Thus in "the death of Caesar," anything else may be substituted for Caesar, but a proper name must not be substituted for death, and hardly anything can be substituted for of. Of the unity in question, the former class of constituents will be called terms, the latter concepts. We have then, in regard to any unity, to consider the following objects:

(1) What remains of the said unity when one of its terms is simply removed, or, if the term occurs several times, when it is removed from one or more of the places in which it occurs, or, if the unity has more than one term, when two or more of its terms are removed from some or all of the places where they occur. This is what Frege calls a function.

(2) The class of unities differing from the said unity, if at all, only by the fact that one of its terms has been replaced, in one or more of the places where it occurs, by some other terms, or by the fact that two or more of its terms have been thus replaced by other terms.

(3) Any member of the class (2).

(4) The assertion that every member of the class (2) is true.

(5) The assertion that some member of the class (2) is true.

(6) The relation of a member of the class (2) to the value which the variable has in that member.

The fundamental case is that where our unity is a propositional concept. From this is derived the usual mathematical notion of function, which might at first sight seem simpler. If \( f(x) \) is not a propositional function, its value for a given value of \( x \) (\( f(x) \) being assumed to be one-valued) is the term \( y \) satisfying the propositional function \( y = f(x) \), i.e. satisfying, for the given value of \( x \), some relational proposition; this relational proposition is involved in the definition of \( f(x) \), and some such propositional function is required in the definition of any function which is not propositional.

As regards (1), confining ourselves to one variable, it was maintained in Chapter vii that, except where the proposition from which we start is predicative or else asserts a fixed relation to a fixed term, there is no such entity: the analysis into argument and assertion cannot be performed in the manner required. Thus what Frege calls a function, if our conclusion was sound, is in general a non-entity. Another point of difference from Frege, in which, however, he appears to be in the right, lies in the fact that I place no restriction upon the variation of the variable, whereas Frege, according to the nature of the function, confines the variable to things, functions of the first order with one variable, functions of the first order with two variables, functions of the second order with one variable, and so on. There are thus for him an infinite number of different kinds