Frege’s goal is to provide an account of the foundations of arithmetic, and to determine whether the propositions of arithmetic are *a priori* or *a posteriori*, and whether they are analytic or synthetic. He will maintain that one can provide definitions of ‘0’ and of ‘the successor of n’ in purely logical terms, and using these one can define each finite numeral; e.g.

\[ 1 =_{df} \text{the successor of } 0 \]
\[ 2 =_{df} \text{the successor of } 1 \]
\[ \text{etc} \]

One can also define ‘is a number’, and ‘is a finite number’ within logic. Using these definitions, one can derive known laws of arithmetic from logical principles alone. (A crucial step in this derivation is to prove that every number has a successor.) He will conclude from this “reduction” of arithmetic to logic that the truths of arithmetic are analytic and *a priori*.

The **Introduction** and Sections 1-4 describe the task of the book.

**Sections 5-8** ask whether numerical formulas are provable. Frege thinks the answer is yes. Kant and Mill say not; their views are criticized. Leibniz says yes, but his proofs contain gaps.

**Sections 9-11** ask whether the truths of arithmetic are inductive – that is, whether they are established by generalizing from experience. (The ‘induction’ referred to here is not what is commonly called ‘mathematical induction’.) Frege thinks not. Mill’s view is especially criticized.

**Sections 12-17** discuss whether the truths of arithmetic are *a priori* or *a posteriori*, and whether they are analytic or synthetic. Frege rejects Kant’s view that they are synthetic *a priori* and Mill’s view that they are *a posteriori*. He agrees with Leibniz that they are *a priori* and analytic, but Leibniz has failed to establish this. He discusses the difference between geometry and arithmetic; geometry deals with space, and requires experience to establish its principles, whereas arithmetic does not.

**Sections 18-20** ask whether the concept ‘number’ can be defined. Frege will later on define it. As a preparation for such a definition, he needs first to determine what kind of thing a number is.

**Sections 21-25** argue that numbers are not properties of external things, and **Sections 26-27** argue that numbers are not subjective.

We will skip Sections 28-44, in which a number of views of other authors are criticized. These sections are interesting, but some are difficult to read because the views being criticized verge on incoherence.

**Section 46** gives Frege’s key insight, which distinguishes his own view of number from those of everyone before him. He suggests that we must pay careful attention to the sentences in which we apply arithmetic to things, such as ‘Four horses pull the king’s carriage’ or ‘There are five books on the table’. Frege holds that each of these makes an assertion about a concept. The first makes an assertion about the concept of *being a horse that pulls the king’s carriage* (the assertion is that four things fall under it), and the second makes an assertion about the concept of *being a book on the table* (that five things fall under it). In sections 47-54, which we skip here, Frege argues that this just-mentioned view is correct. (A concept for Frege is not something mental; roughly, a concept is what others will call a property.)

One might think from the discussion so far that Frege will decide that numbers are properties of concepts, namely, the properties that are asserted of the concepts in the examples above. However, he thinks that numbers are in fact objects, not properties, and when we attribute to the concept of *being a book on the table* the property *that five things fall under it*, that property is not itself the number five; rather, the
number five forms part of the property. Frege sees that property as consisting of a relation between a number and a concept; this relation he calls “belonging to”. So ‘There are five books on the table’ will be analyzed as ‘the number five / belongs to / the concept of being a book on the table’. His task will be in part to define ‘the number five’ and ‘belongs to’ using purely logical terminology. (He will presuppose that ‘the concept of such and such’ is a basic notion of logic, and does not need to be defined.)

Section 55 gives some provisional “definitions”, which, though correct, will not work as definitions; section 56 explains why these provisional definitions are not adequate as definitions.

Section 57 explains why Frege thinks that numbers are objects, and not properties.

Sections 58-61 discusses what numbers as objects are like; e.g. they are not things we have ideas of, and they do not exist in space.

Sections 62-69 develops his account that “the number which belongs to concept F is the extension of the concept equal to the concept F”.

Sections 62-63 show how to define “the number which belongs to the concept F is the number which belongs to the concept G”.

Sections 64-65 give examples of a similar account of the “direction” of a line. We are not trying to define ‘is’ (‘is identical with’); that is already a term of logic; we are trying to define ‘belongs to’. Sections 66-67 explain why we still have not arrived at a definition.

Section 68 finally gives Frege’s definition: The number which belongs to the concept F is defined to be the extension of the concept being equal to F.

We are skipping sections 70-83 which fill in the rest of the technical account of arithmetical notions. They go roughly as follows:

Frege defines the relation of two concepts being equal (in size) in sections 71-72.
He has previously defined the relation belonging to: n belongs to F iff n is the extension of the concept equal to the concept F
He defines ‘n is a number’ as ‘there exists a concept which n belongs to’.
0 is defined to be the number which belongs to the concept not identical with itself
(so, 0 belongs to a concept under which nothing falls)
1 is defined to be the number which belongs to the concept identical with 0
(so, 1 belongs to a concept under which exactly one thing falls, namely, 0)
The successor relation is defined as:
n directly follows m (in the series of natural numbers) iff there is a concept F and an object x falling under F such that n belongs to F and m belongs to the concept falling under F but not identical to x
(So n follows m iff n belongs to a concept which applies to one more thing than the concept that m belongs to. One can then prove, for example, that 1 (directly) follows 0.)
A natural number (finite number) is defined to be anything that (directly or indirectly) follows 0.
Finally, Frege sketches a proof that after every natural number there follows another.

We are skipping sections 84-86, which discuss infinite numbers, and sections 87-104, which discuss a number of philosophical, logical, and mathematical matters.

Sections 106-109 are Frege’s own summary of what he has accomplished.