1879 *Begriffsschrift* (*Conceptual Notation*)
Contains a formulation of the higher order predicate calculus with identity.
Contains a complete set of axioms and rules for the first-order part.
Shows how to define ‘x follows y in the R series’.
[Roughly, it goes: x is in every set which (i) contains y and (ii) whenever it contains u, and R relates u to v, it also contains v.]

1884 *Foundations of Arithmetic (Die Grundlagen der Arithmeitik):*
A popular work intended to state his program of reducing arithmetic to logic and to criticize alternative views about the foundations of arithmetic.

1893, 1903 *Basic Laws of Arithmetic (Grundgesetze der Arithmetik):*
An attempt to “put it all together” working out the necessary details.
[Ultimately, the project of reducing arithmetic to logic failed, though a great deal was learned along the way. Frege’s formulation of logic is accepted today (in modern notation). Frege’s semantic theory that he formulated as part of his overall view remains of great interest.]

**Question #2.** The logical and epistemological status of arithmetic truths.

\[ a \text{ posteriori} \equiv df \text{ not } a \text{ priori} \]
\[ \text{synthetic} \equiv df \text{ not } \text{analytic} \]

Kant: \( a \text{ priori} \) = can be known independent of any experience
Frege: \( a \text{ priori} \) = can be proved without any appeal to facts

Note that a proposition may be \( a \text{ priori} \) when it could not be known without experience, if, e.g., you need experience to even understand it. The “independent of any experience” addresses the rationale of the knowledge, not its source. Frege clarifies this by making the rationale be in terms of proof.

Kant: \( \text{analytic} \) = the predicate is already contained in the subject
Frege: \( \text{analytic} \) = can be proved from general logical laws and definitions.

Two problems with Kant’s account of analyticity:
1. Containment is not clarified. E.g. is ‘Every giraffe is spotted’ analytic?
2. The explanation gives the wrong answer for analytic negative sentences, such as ‘No circle has corners’, and does not apply at all to sentences which are not of subject-predicate form, such as ‘Every dog is spotted or not every dog is spotted’.

Frege avoids these problems by referring instead to general logical laws, but he does not say what a general logical law is. However, he does give a fairly complete account of which principles are logical.

Frege’s own view: arithmetical propositions are a priori and analytic.
**Kant's views:** [Kant does not discuss general truths of arithmetic, such as “n+m = m+n”.] Kant holds that particular equations (7+5 = 12) are unprovable and synthetic, but *a priori*.

Frege's Objections to Kant’s view:
- It is incongruous and paradoxical for there to be an infinite number of unprovable primitive truths.
- Such equations are not self-evident when the numbers are large. The only way we can recognize them to be true is to prove them.
- Kant holds that they are based on intuitions of fingers or points, but we have no such intuitions when the numbers are large.
- Equations with large numbers must be provable; then why not equations with small numbers?

**Leibniz's views:** Numerical formulas are logically provable by substituting definitions for the terms they define, and vice versa. E.g. given the definitions: 2 =df 1+1; 3 =df 2+1; 4 =df 3+1;
we can prove 2+2 = 4 as follows:

- **[line 1]:** 2+ 2 = 2+ 2  Since anything is identical to itself
- **[line 2]:** 2+2 = 2 + 1+1  Replacing the last ‘2’ by its definition which is ‘1+1
- **[line 3]:** 2+2 = 3 + 1  Putting 3’ for its definition, which is ‘2+1’
- **[line 4]:** 2+2 = 4  Putting ‘4’ for its definition, which is ‘3+1’

However, Frege points out that the proof has a gap:
- In order for **[line 2]** to be justified the right-hand side must be read as ‘2 + (1+1)’
- In order for  **[line 3]** to follow from **[line 2]**, the right-hand side of **[line 2]** must be read as ‘(2+1)+1’.

Leibniz assumes that grouping does not matter, so that (2+1)+1 = 2+(1+1). But this needs to be proved, not assumed. {Notice that grouping does matter for subtraction: (2-1)-1  2-(1-1).}

Note: For Frege, it is OK for each numeral to be defined in terms of its immediate predecessor. That is, he will define a relation of “successor”, and prove that each natural number has a unique successor; then one can give the definitions:

- 2 = the successor of 1
- 3 = the successor of 2
- 4 = the successor of 3

One can then later on define addition, and prove that the successor of n is n+1. But these details are not discussed in *FA*.

**Mill's views:** Arithmetical truths are empirical (synthetic), *a posteriori*. Definitions of each numeral as being the successor of the previous numeral are empirical generalizations. An example: “3 = 2+1” means that things with impress our senses as a group of three things near one another can be separated into a group of two of them near each other, and a third one not near those two.

Frege’s objections to Mill:
§7: If objects could not be moved, then 3 would not be 2+1. This is absurd. This kind of account cannot be given of 0 and of 1.
If this is right, we cannot speak of three strokes of a clock, or three sensations, etc.
Mill himself gives Leibniz’s proof that 2+2 = 4, but never says where the empirical generalizations should come in.

Mill says that complex mathematical truths are not based directly on observation, but are instead derived from a general law established by induction. But what is that law? How is it empirically established?

§8: Mill thinks we can construct large numbers out of smaller ones. But how?

§9: Mill thinks that 1=1 might be false, since one pound weight does not always weigh the same as another. But ‘1=1’ does not say that it does. Mill confuses arithmetic truths with their applications. E.g. he wrongly equates heaping up with addition. He mistakenly thinks that ‘+’ expresses a relation between the parts of a thing and the whole.

§10: If mathematical truths are based on induction, what observations do we start with? Mill does not say.

Induction depends on observing things and assuming that other things will be the same. But each number is unique. Suppose you try to accommodate this fact by holding that you can deduce the unique properties of a number from those of its predecessor and the fact that it is one greater? This appeals to definitions – it makes sense on Frege’s approach, but not on Mill’s.

Induction is based on probabilities, but the use of probability presupposes truths of arithmetic.

**Question #3**: Why is number not a property of external things, and why are numbers not subjective?

**Is number a property of external things?**

Frege considers two cases:

- Why numbers are not properties of external objects
- Why numbers are not properties of agglomerations of external objects

§22: Compare:

- The tree has 1000 leaves
- The tree has green leaves

The latter attributes greenness to each leaf, but the former does not.

§23: Attacks Mill’s view that “A number ascribed to an agglomeration is the characteristic manner in which it is made up of, and may be separated into, parts.” Objections:

- There is in general no such unique way. E.g. a bundle of straw may be separated into two half bundles, into individual straws, cut into half-straws, etc.
- The number 1 fails to do justice to ways in which a thing may be split up.
- Things need not be together (unseparated) in order to number them.
- What about zero?
- We can number anything, even things without parts, such as immaterial things.
- We don’t find the same sensible thing in nonsensible things. If we abstract 3 from the impression of a triangle, how could this be applied to three concepts?

**Is number something subjective?**

§26: Do not confuse the ideas you get when thinking of something with that thing itself. ‘Objective’ means what is independent of our sensation, intuition, imagination, and all construction of mental pictures. Arithmetic truths are objective in this sense. Truths about ideas are not.

Mental processes are never appealed to in proofs.
§ 27: If numbers were mental, we would have millions of twos – yours, mine, etc. We would have to speak in the plural about twos. They would evolve over the generations. We would also be doubtful whether very large numbers exist.
Question #4: Discuss Frege’s own definition of number.

§46: “...consider number in the context of a judgment which brings out its basic use.”
“...the content of a statement of number is an assertion about a concept.”
Example: ‘Venus has 0 moons’ means ‘0 belongs to the concept *moon of Venus*’

<Note that concepts are like agglomerations which have unique modes of division.
The concept *book on the table* uniquely divides into individual books.>

The content of a statement of number is an assertion about a concept:

\[
\text{Four horses draw the king’s carriage} \approx [\text{Applies to four things}]: [\text{the concept of being a horse that draws the king’s carriage}]
\]

Maybe this is the number 4? Not for Frege:

§56: “In the proposition ‘the number 0 belongs to the concept F’, 0 is only an element in the predicate. ... For this reason I have avoided calling a number such as 0 or 1 or 2 a property of a concept.”

The number four is only an element in the predicate (in “applies to four things”):

\[
\begin{array}{ccc}
4 & \text{belongs to} & [\text{the concept of being a horse that draws the king’s carriage}] \\
\downarrow & & \downarrow \\
\text{an object} & \text{a relation} & \text{a concept} \\
& \text{between objects} & \\
& \text{and concepts} & 
\end{array}
\]

The task for Frege:

The number-objects need to be explained.
The relation of belonging to needs to be explained.
Concepts do not need explanation; they are a part of logic.

§55: It would be tempting (but wrong) to define:

\[
\begin{align*}
0 \text{ belongs to } F & \equiv_{df} \forall a: a \text{ does not fall under } F \\
n+1 \text{ belongs to } F & \equiv_{df} \exists a: a \text{ falls under } F \text{ and } n \text{ belongs to the concept } \text{“falling under } F \text{ but not being } a\text{”}
\end{align*}
\]

Frege thinks that these statements are true. But they cannot be definitions. Because these “definitions” do not let us know the sense of the expression “the number n belongs to the concept G”. For example,

1. The definitions do not let us decide whether the number Julius Caesar belongs to any concept.
2. The definitions do not let us prove that only one number belongs to a concept.
§65-68: A new kind of definition:

We want to define what a direction is by exploiting the equivalence between:

- line \( a \) is parallel to line \( b \)
- the direction of line \( a \) = the direction of line \( b \)

This is accomplished by defining:

- the direction of \( a \) =_df the extension of the concept: being parallel to \( a \)

Frege’s own definition of belonging to:

- the number which belongs to \( F \) =_df the extension of the concept: “equal to \( F \)”

(where “equal to \( F \)” has already been defined. A concept \( G \) is equal to \( F \) iff there is a 1-1 relation between the \( G \)’s and the \( F \)’s.)

Frege’s definition is meant to be a definition of ‘belongs to’, not ‘being a number which belongs to’, because the definition actually says nothing about numbers. That is, the definition is better formulated as:

- \( x \) belongs to \( F \) =_df \( x \) is the extension of the concept: “equal to \( F \)”.

Then in §72 Frege will define “is a number” as:

- \( x \) is a number =_df \( x \) belongs to some concept]

Then he will show that these definitions work as we hope.

How do we know that Julius Caesar is not a number? This question is not discussed further.

How do we know that a concept has only one number belonging to it? This can be proved from the logical principles that Frege assumes, though he does not prove it here.
**Question #1.** The three fundamental principles from the Preface are:

1. Separate the psychological from the logical; the subjective from the objective
2. Never ask for the meaning of a word in isolation.
3. Distinguish concepts from objects.

1. **“Separate the psychological from the logical; the subjective from the objective”**
   This question is addressed in connection with Question #3. Note that none of Frege’s objections depend upon there being anything wrong *per se* with psychological events or objects, or their study, or with subjective truths. It just so happens that arithmetic deals with logical and objective things, not psychological and/or nonobjective.

2. **“Never ask for the meaning of a word in isolation.”**
   Frege’s stated reason is that if you ask for the meaning of a word in isolation, you invite an account of what “comes to mind” when someone ponders with the word. But what comes to mind are ideas associated with the word. These are not the word’s meaning, nor do they tell you what it stands for, if anything.

   Another reason, not explicitly mentioned here by Frege: He thinks that words come in various types; some stand for objects, some for concepts of objects, some for concepts of concepts of objects, etc. In order to see what role a particular word has, you have to see how it functions in a sentence. For example, it is a matter for investigation what the connection is between the meanings of number adjectives in ordinary talk, as “four apples”, and of the numerals dealt with by arithmetic, such as “two + two = four”.

   [Frege held later that concept-words have no meaning in isolation. This view does not seem pertinent to issues in *FA*. And he never says that object-words have no meaning in isolation.]

3. **“Distinguish concepts from objects.”**
   This view was developed in detail by Frege over the years. His most famous views are given in “On Concept and Object” (1892) where he holds that concept words are “incomplete” or “unsaturated” or “predicative”, whereas sentences and names of objects are not. This mature view does not seem to be featured in *FA*. Instead, the slogan enters into *FA* in two ways.

   (i) Frege’s stated reason for making this distinction in *FA* is that this will let you see what is wrong with popular *formalist* views. This is not a topic covered in our reading.

   [It goes something like this. The formalists think that just by defining a term you thereby give it a content. So e.g. you can give content to “the square root of −1”. They then think that this shows that there is such a thing as the square root of −1. (Or, it shows whatever amounts to that for the puroposes of arithmetic.) Frege thinks that by defining a concept-word (such as “__ is a square root of −1”) you thereby insure that it stands for a concept, but by defining an object-word (such as “the square root of −1”) you do not thereby insure that it stands for an object, and the formalists think you do because they don’t distinguish concepts from objects. Frege thinks that for purposes of arithmetic you need to be sure that a word such as ‘2’ actually stands for something -- for an object.]

   (ii) The second reason arises in section 57, where it figures essentially in his reasoning about the analysis of attributions of numbers to concepts. Such attributions must be complex, because the number part is an object, and we need to supply more to predicate the object of the concept.