The note below contains an almost inexplicable mistake. Where I say, "Distinct concepts may be coextensional (having a heart and having a liver)", I should have included a "not" between "may" and "be". My mind wandered, and I was thinking of Russell. For Frege, a concept is a function from individuals (or other concepts) to truth values, and I myself believe that for Frege, coextensional concepts are identical. I actually argued this case, in tedious detail and at tedious length, in a recent paper. Those of you with nothing better to do can take a look at Sections 1.2.2-1.2.5 in the attached, especially footnote 72 on the 25th pdf page (page 957 in the document page numbering).

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This is just about what I said. Good work, Gabe!

I want to emphasize and clarify a few small details.

First, Terry spoke about, and you were to read from, Frege's Grundlagen (Foundations of Arithmetic). The stuff about Cantor and Axiom V is derived from Frege's Grundgesetze (Basic Laws of Arithmetic). As Terry said, Grundlagen outlines the program and includes polemical and philosophical remarks. Grundgesetze is a technical work that aims to carry out the program in exact detail. The latter contains additional technical details that could not be precisely predicted from the former. It also includes as a new idea, the introduction of Axiom V.

There are two German words "Menge" and "Klasse", which are usually translated set and class. Cantor called his set theory "Mengenlehre". Frege used "Klasse" for when he characterized the extension of a concept as a class. He took this to be a logical idea, and he took Cantor's theory to involve (roughly) a new category of mathematical entities. In a history of logic course you may be told that the very idea of the entities that are generated as the extensions of predicates or properties by a Comprehension Axiom like EyA(x e y iff Phi x) is a different idea from the idea of the entities that are generated from the empty set by iteration of the operation of forming the set of all subsets. The former would be something like Frege's Klassen, the latter something like Cantor's Mengen.

Cantor's argument, as presented by Russell in the language of Grundgesetze, poses a challenge to both forms of the theory of sets/extensions. In Frege's theory, Axiom V says that two concepts correspond to the same object iff the same objects fall under them (i.e. that are co-extensional). (You can see why Frege introduced these objects as the extensions of concepts since they are postulated as the common object corresponding to coextensional concepts.) Distinct concepts may be coextensional (having a heart and having a liver), but there will still be an object corresponding to every subset of objects, and this is, as Cantor showed, too many.

There can be no objection to defining a relation of co-extensionality between concepts F and G when Ax(Fx iff Gx). The problem arises when we postulate that there is an object (among the values of the variable "x") that constitutes the extension of each concept. Axiom V seems quite natural from the point
of view that postulates such objects.

At 12:00 PM 10/2/2006, Rabin, Gabriel wrote:

Dear Classmates,

After class on Thursday, David gave a nice little spiel about Frege’s general view, elucidating his framework of concept levels. After he finished, I told David that sharing this information with the whole class, rather than just with those of us who happened to be in the room at the time, would be help us understand the material. David agreed, and suggested that I write up something informative. While this was not the result I was aiming for, I will do my best to share with you all what David clarified to a few of us.

Frege believes in both concepts and objects. They can be arranged in a hierarchy. Objects, such as David or his ball point pens, are on the bottom level. There are many concepts, and an unlimited number of levels to these concepts. 1st level concepts, such as “blue”, are concepts of objects. 2nd level concepts, such as “color”, are concepts of 1st level concepts. The 2nd level concept “color” applies to the 1st level concept “blue”, which in turn applies to the object “David’s pen”. Every concept has a level, and every concept applies to those entities (concepts or objects) on the level below it. Every concept also has an extension. This extension is an object, on the first level. This is an interesting doctrine (more on this later).

What are numbers? Well, according to Frege, a number x belongs to a concept F iff x is the extension of the concept “equinumerous with F”. So the number 1 belongs to a concept F iff 1 is the extension of the concept “equinumerous with F”. A concept G will be equinumerous with F if and only if G and F are true of the same number of entities. If 1 belongs to F then F is true of only one thing. G is equinumerous with F iff G is true of only one thing, i.e. 1 belongs to G. What is the number 1 itself? It’s an extension, which is an object. It’s the extension of the concept “equinumerous with [insert concept that applies to only one thing here]”. If we allow ourselves to thinks of extensions as sets, then 1 looks like the set containing either: (i) all the concepts that are true of only one thing or (ii) all the one membered sets.

Frege’s doctrine of extensions creates a correspondence between concepts and objects. For every concept there is a corresponding object, its extension. This leads Frege into trouble. Mathematicians (Cantor?) demonstrated that the number of subsets of a set is greater than the number of elements in the set. If there is a concept for each subset of the set of bottom level objects (we can easily imagine the disjunctive concept “is object 1, or object 2, or … or object n”), then there are more concepts than objects. But according to Frege, for every concept there is a corresponding object, its extension. Paradox can be avoided if more than one concept can have the same extension. But Frege’s Law V, along with his identity conditions for concepts, prevents this scenario. If two concepts have the same extension they are the same concept. This leads to the paradox of Frege’s logical system from the Grundgesetze. We can abandon Law V, or we can maintain that not every set of objects has a corresponding concept. Unsavory options.

I hope that my explanation didn't increase confusion and that I didn't say too many incorrect things. See you Thursday.

-Gabe