Frege postulated that every property (which he took as a function from individuals to truth values and which he represented by a predicate or open formula) had an “extension” that was an individual. These extensions (or “value ranges” as he called them) seemed to be something like the collection of individuals that had the property. At any rate, properties had the same extension if and only if they applied to the same individuals. He wrote the following axiom to characterize this new category of individuals.

Axiom 5 \( \forall F \forall G[\forall x(Fx \leftrightarrow Gx) \leftrightarrow \text{ext}(F) = \text{ext}(G)] \)

As Russell discovered, this axiom leads to a contradiction. He formulated the property \( R \) which holds of just those individuals that are extensions of properties they don’t have. Remember, these extensions are individuals, and like other individuals, they have certain properties and lack others. And note that there are properties \( S \) whose extension does not have the property \( S \). Consider the property of not being self-identical. It has some individual as its extension. But no individual can have that property. Russell’s \( R \) applies to all only such properties.

Let \( R \) be the Russell property.
\( R_{\{1\}} \) abbreviates \( \exists F(\{1\} = \text{ext}(F) \land \neg F_{\{1\}}) \)

We can now ask: Does the individual \( \text{ext}(R) \) have or lack the property \( R \)?

The answer is that using Axiom 5 we can derive a contradiction from both answers:

Assume that \( \text{ext}(R) \) has the property \( R \).
I) \( R_{\text{ext}(R)} \)
which in unabbreviated form is;
II) \( \exists F(\text{ext}(R) = \text{ext}(F) \land \neg F_{\text{ext}(R)}) \)
See if you can derive a contradiction from I), II), and Axiom 5.

Now, assume that \( \text{ext}(R) \) lacks the property \( R \).
III) \( \neg R_{\text{ext}(R)} \)
which in unabbreviated form is;
IV) \( \neg \exists F(\text{ext}(R) = \text{ext}(F) \land \neg F_{\text{ext}(R)}) \)
See if you can derive a contradiction from III), and IV).

D.K. 11/20/12