Points About Denoting [1903]

This paper consists of a set of rough manuscript notes which was used in preparing Paper 14. The manuscript is undated, but the list of fundamental notions of mathematics given in the paper are those employed in the functional theory of the later half of 1903. It is unlikely that the paper was written later than April 1904, for reasons given in the headnote to 14. Among the topics of the paper are: knowledge by acquaintance and knowledge by description, the constituents of propositions, and the nature of the variable. The latter half of the paper is devoted to discussing the formulation of quantification theory in the light of the basic notions considered earlier. As in Papers 11 and 14, Russell assigns both meaning and denotation to complexes.

The copy-text is the manuscript (RA 220.010960), which consists of eighteen leaves; folio 16 was transferred to Paper 14 as folio 49.
(1). That sometimes we know that something is denoted, without knowing what.

This occurs in obvious instances, as e.g. if I ask: Is Smith married? and the answer is affirmative, I then know that “Smith’s wife” is a denoting phrase, although I don’t know who Smith’s wife is. We may distinguish the terms with which we are acquainted from others which are merely denoted. E.g. in the above case, I am supposed to be acquainted with the term Smith and the relation marriage, and thence to be able to conceive a term having this relation to Smith, although I am not acquainted with any such term.

So again in other cases. It is known that every class of material points has one centre of mass: this is demonstrated without reference to particular classes of material points. Thence, given a particular class, e.g. the Solar System, we infer that it has one centre of mass; thus we can denote the centre of mass in question, without being acquainted with it.

Generally speaking, we may know, without leaving the region of general propositions, that every term of the class a has the relation R to one and only one term; as e.g. we know that every human being now living has one and only one father. Thus given any term of class a, say x, we know that “the term to which x has the relation R” has a perfectly definite denotation. Nevertheless, it’s a wise child etc. This shows that to be known by description is not the same thing as to be known by acquaintance, for “the father of x” is an adequate description in the sense that, as a matter of fact, there is only one person to whom it is applicable.

(2). When a denoting phrase occurs in a proposition, does that which is denoted form a constituent of the proposition or not?

Recurring to Smith’s wife, let us baptize her Triphena. Then “Triphena is Smith’s wife” is a significant proposition, but “Smith’s wife is Smith’s wife” is a tautology. Thus it would seem that Triphena is not a constituent of the latter, for if she were, there could hardly be any difference of the two propositions. Nevertheless “Smith’s wife has blue eyes” is a statement about Triphena. Hence a difficulty.

In this matter, I suggest the following compromise. In a complex, we must distinguish the meaning and the denotation. If the meaning is complex, the whole is called a complex, although the denotation may be simple. Constituents of the meaning of a complex may not be constituents of the denotation, and vice versa. Thus in “Smith’s wife has blue eyes”, I should say that Smith and wife and the meaning (not the denotation) of “Smith’s wife” are constituents of the total meaning, but none of them are constituents of the denotation, whereas Triphena herself is a constituent of the denotation. But if I say “Triphena has blue eyes”, then Triphena is a constituent both of the meaning and of the denotation. This raises curious points with regard to identity: two complexes may be iden-
tical in denotation, and different in meaning, though the converse is apparently impossible. Thus “Triphena has blue eyes” and “Smith’s wife has blue eyes” have the same denotation, but different meanings. The same applies to “Triphena is Smith’s wife” and “Smith’s wife is Smith’s wife” and “Triphena is Triphena”. A proposition will only be said to be about a term if that term is a constituent of the denotation. Thus “Smith’s wife has blue eyes” is about Triphena, but not about Smith. On the other hand, “Smith married a woman with blue eyes” is about Smith, but not about Triphena, whereas “Smith married the only daughter of Ebenezer” is about both Smith and Triphena.

(3). It would seem that every complex either is or presupposes a proposition; it is possible to maintain that when this proposition is false, the complex has only meaning and no denotation.

A complex of the form “the \(u\)” involves the proposition that \(u\) is a concept of which there is one and only one instance. When this proposition is true, the complex denotes the said instance; when false, it denotes nothing. Thus “The King of England” denotes Edward VII, but “the King of France” denotes nothing. But what shall we say of such instances as “the difference between \(a\) and \(b\)”? It would seem that (taking difference in the sense of bare diversity) what is denoted (in the case when \(a\) and \(b\) do differ) is the same as what is denoted in the proposition “\(a\) and \(b\) differ”. We must not attempt to define the truth or falsehood of propositions by reference to denoting, for this would lead to a vicious circle. But it seems consonant to common sense to hold that a true proposition denotes a fact, while a false one denotes nothing.

I am doubtful as to whether such complexes as “not-\(u\)” involve any proposition at all—meaning by not-\(u\) the negation of the predicate \(u\), not that which is diverse from \(u\). But certainly most complexes involve propositions, and this case may be left open.

(4). We must distinguish between not denoting anything and denoting the entity nothing.

Hitherto, when I have spoken of denoting nothing, I have meant not denoting anything; the other is best spoken of as “denoting nothing”, where the italics avoid ambiguity. Nothing itself is a class of individuals, but a class containing no members. There are many difficulties regarding it, but they need not be considered in this connection.

(5). It is necessary, for the understanding of a proposition, to have acquaintance with the meaning of every constituent of the meaning, and of the whole; it is not necessary to have acquaintance with such constituents of the denotation as are not constituents of the meaning. It is thus that definitions proceed. They give a known meaning, and enable us to make propositions about what the meaning denotes, although we may have no acquaintance with this something. In the proposition “there is one and
only one instance of \( u \), the said instance is not a constituent of the meaning, hence the proposition may be known without our being acquainted with the instance. Thence we can define the instance as “the \( u \), and make propositions about it, while yet remaining unacquainted with it. But in such propositions, the instance in question will never be a constituent of the meaning, but only of the denotation.

(6). The relation of functions to denoting is difficult.

If we consider such a case as “Smith’s wife”, we seem to have the following functions: (\( \alpha \)) \( x \)'s wife, (\( \beta \)) Smith’s \( y \), (\( \gamma \)) \( x \)'s \( y \). Each of these must be twofold, according as, when a value is supplied to the variable, the value of the function is to be the resulting complex meaning, or what this meaning denotes. It is plain that the latter is what is usually in question, for if \( x \) and \( z \) have the same wife, we should say the function (\( \alpha \)) has the same value for \( x \) and \( z \), which is only true of the denotation.

Consider the more general case, e.g. \( xRa \). The function involved is to be such that, for every value of \( x \), the value of the function is that which is denoted by \( xRa \). If we regard the function as being “a term having the relation \( R \) to \( a \)”, we must regard this as having both meaning and denotation; the denotation is the function.

(7). The fundamental point is this: If two complexes are different in meaning but identical in denotation, they are identical, and not two but one: meaning has to do, not with what a thing is, but with the road by which it is reached: in a denoting complex, the complex apart from what it denotes may be not a single entity at all.

The fundamental notions in mathematics are

\[
(x) \cdot p \quad x(p) \quad p^y_x \quad \phi|x.
\]

In the first three, \( p \) is to be supposed a dependent variable, though of course the case where it is not so must be provided for.

Df. (not in the strict symbolic sense). A dependent variable is something denoted by a complex of which the independent variable in question is a constituent.

(\( x \) \( \cdot p \) means: “the truth of \( p \) for every value of \( x \)”. If \( p \) is not a variable dependent on \( x \), this is the same as “the truth of \( p \)”; if \( p \) is such a dependent variable, (\( x \) \( \cdot p \) means that however \( x \) may vary, \( p \) is always a true proposition.

\( p^y_x \) means “that which is denoted by the meaning which results from giving \( x \) the value \( y \) in \( p \)”. The difficulty of this is, that we may have \( p = q \) and not \( p^y_x = q^y_x \).
E.g. The Prime Minister of England = Arthur Balfour
(The Prime Minister of England) England = M. Combes

What we want is \( x . p = q \), whence \( p^y_x = q^y_x \) does follow.

The point to be brought out is this: The variable, both dependent and independent, is a fundamental notion, not capable of further analysis. If \( X \) be a dependent variable, and \( x \) its independent variable, there is a nexus of \( X \) and \( x \), which is not determinate when single values of \( X \) and \( x \) are given, but becomes determinate when the *function* involved is given. If the *function* is given, that amounts to giving the nexus of \( X \) and \( x \). Now this nexus may be conceived in two ways, \((\alpha)\) as the bare scheme of the complex meaning involved—*i.e.* roughly, as the complex meaning with the variable left variable—or \((\beta)\) as the correlation of the independent variable with the term denoted by the complex meaning concerned. The former is intensional, the latter extensional. The extensional point of view is always preferable in mathematics. Thus given the scheme of a complex meaning, we shall regard this as denoting a certain correlation, and the correlation will be what we shall call the function. Thus the complex meaning may be changed without changing the function, provided the correlation remains unchanged.

But \( p^y_x \) still presents difficulties. If \( p \) stands for the complex meaning, \( p^y_x \) should stand also for a complex meaning, not for what this denotes.

If \( p \) stands for what the complex meaning denotes, then \( p = q . \bigwedge . p^y_x = q^y_x \), in fact, the *meaning* of \( p \) as opposed to the denotation ought to be irrelevant, which it isn’t. If we make \( p \) stand for the correlation, \( p \) has become a function, not a dependent variable, and we run against the Contradiction, since not all dependent variables lead to functions. The fact seems to be that \( x \), the independent variable, is an indefinable, and \((\text{e.g.})\) “the father of \( x \)” is something quite definite, distinct from all its values, and from the function involved, and from the *meaning* of the complex “the father of \( x \)”. Now \( p^y_x \)—e.g. (the father of \( x \))^\(y\)\(\_x\)—is a new dependent variable, containing two independent variables. But if \( y \) is a constant, then \( p^y_x \) is a *value* of \( p \), which is quite a different thing. What results is the absolute distinctness of a variable from everything else.
The \( \text{Pp} \) to which Whitehead objects is

\[
\vdash: \text{Cls} | u: (x). \text{Prop} | p: (x). u x \equiv p \supset u = x(p) \quad \text{Pp}
\]

\textit{i.e.} if the value of \( u \) is always a proposition, and \( p \) is always a proposition equivalent to the corresponding value of \( u \), then \( u \) and \( x(p) \) are identical. Thus \textit{e.g.} Being a man = \( x \) is a featherless biped). Certainly it seems evident that, if we take an extensional view of functions, to be a man is the same as to be a featherless biped. We want to say that "\( x \) is a man" has the same denotation as "\( x \) is a featherless biped", \textit{i.e.} the fact denoted is the same. But certainly this seems doubtful; and it is better to avoid this assumption if it can be dispensed with. Nevertheless, it is obvious that "a man" extensionally is identical with "a featherless biped". This seems again to justify the assumption.

The chief new point to be borne in mind is the fundamental nature of the variable. This was already involved really in the Calculus of Propositions, \textit{e.g.} in \( \ast 2.31 \). We are supposed to be able to assert \( \phi | x \) as opposed to \( (x) \cdot \phi | x \), and this requires the recognition of the variable as fundamental.

The Propositions of \( \ast 2 \) are now as follows:

\( \ast 2.2 \quad \vdash: (x). p \supset p \frac{a}{x} \)

\( \ast 2.3 \quad \vdash: (x). p \supset p \frac{y}{x} \)

\( \ast 2.4 \quad \vdash: (x). p \supset q \supset p \supset (x). q \)

But \( \ast 2.31 \) has become impossible to state. It was \( \vdash. \phi | x . \supset \vdash. (x) . \phi | x \). But this won't do, because it uses a function where we want a complex. The form

\[ \vdash. p . \supset . (x) . p \]

will do: for if \( p \) does not contain \( x \), \( (x) . p \) is the same as \( p \). But it must be clearly understood that \( x \) is variable. We don't have

\[ \vdash: (x): p . \supset (x) . p \]

for here, the \( x \) in \( p \) might have a particular value. The fact is, a complex containing a variable is a whole, and in any part of the complex the variable is not properly independent.

It is the variable thus involved that must be used in explaining complexes.

We want a symbol for "the truth of \( p \) for any value of \( x \)". I doubt if the truth for all values is required; but probably it is.
The above statement of *2·4 won't do; for we require that \( p \) should not contain \( x \).

If we introduce a special symbol for a variable dependent variable, the implications involved will only hold when the symbol has a dependent variable for its value. In other cases, they will be false or meaningless. What \( \phi x \) does, is to analyze the dependent variable into a constant complex of two independent variables; but this can't always be done. The only way I can see of re-stating *2·4 is

\[
*2·4 \quad \vdash: (x). p \supset q \; \supset: (x). p \cdot \supset: (x). q
\]

But then we shall need \( \text{Pp}'s \) to show that

\[
x \sim \text{var} \; \supset: (x). p \cdot \equiv \cdot p
\]

and this will make the whole thing hopeless.

The question is this: Given an independent variable \( p \), are dependent variables containing \( x \) among its possible values? If not, \( p^y_x \) won't do. Now the general view of an independent variable is that all its values are constants; thus only the value of a dependent variable for a given value of \( x \) could be included among the values of \( p \). There would have to be a second-order variable, whose business it is to take all values that are complexes containing \( x \). This is exactly what is usually represented by \( fx \) or \( \phi x \). In that case, the \( f \) or \( \phi \) does have a constant meaning, as giving a value to the second-order variable. But the \( f \) or \( \phi \) must not be taken alone.

If the independent variable is anything definite, how comes it that there can be many independent variables? This occurs, when it does occur, only in complexes. In these, when a certain position is filled by a variable, it may happen that, though the variable may be any term, it must be the same term as in a certain other position; or this may not happen. This is how several independent variables are possible.

Consider e.g. \( p \supset q \). This contains two variables. The attitude is, at present, that there is such an object as "\( p \supset q \)"; distinct from all the values obtained by giving definite values to \( p \) and \( q \). Similarly, and more fundamentally, there is such an object as \( p \), distinct from all the values of \( p \). But the difficulty lies in this: \( p \) is not a definite object, for if it were, it would be the same as \( q \); \( p \) is any term, and \( q \) is any term, and each is merely and solely any term, and yet they are not identical.

We must return to the usual usage of \( fx \), \( \phi x \) etc. for a variable dependent variable, leaving it open under what circumstances \( f \) and \( \phi \) can themselves be separately considered. If \( x \) is genuinely a variable, \( p \) can't stand for a dependent variable containing \( x \), but only for some value of such a dependent variable. Thus \( p^y_x \) and the rest must be abandoned. It must be
an essential principle of the symbolism that no expression depends on any variable which does not appear in it.

The plan now is to use $fx$ or $\phi x$ for an unsubanalyzed complex, and to write $f|_x$ or $\phi|_x$ in the case where we can detach a constant element, to be called the function. But it may be questioned whether this is now worth while. Perhaps it is better and simpler to recur to Werthverläufe, writing $f'(fx)$ for the class involved in $fx$. We shall hold that a class is only defined in the case of what we called functional complexes, thus keeping the detail of the work unchanged. The Werthverlauf will be the object denoted by the constant part (if any) of $fx$.

Observe that $fx$ or $\phi x$ is not itself an entity, any more than $x$ is, though for every value of $x$ and of $f$ or $\phi$ it is an entity.

We shall have to admit and point out certain lacunae in the initial parts of the formalism—these seem not at present avoidable.

The necessity of admitting the dependent variable may be seen thus: we want to make general statements about anything that varies as $x$ varies—e.g. in $\ast 2\cdot 2\cdot 3\cdot 31\cdot 4$. Here, if we take a complex containing $x$, and give $x$ a particular value, we get a value for the complex which is certainly also the value of other complexes for the same value of $x$. Consequently, if we now proceed to vary $x$, we don't get what we want. The $x$ in the complex must be a variable $x$, and the complex must be given by its dependence on $x$, not by its value for a particular value of $x$. Consequently, where we are dealing with a variable complex containing $x$, the particular value of our variable complex is itself still a variable, but a dependent variable, varying as $x$ varies. Now if we take any single letter $p$, a particular value of $p$ may well be the value of some complex containing $x$ for a particular value of $x$, but cannot be the complex with $x$ still variable. And this shows that $p$ or $p^y_x$ won't do, since it forgets that a variable $x$ is not any of its values: it rests on too static a theory of the variable. The fact is that a dependent variable depends upon its $x$ in a way in which its values do not depend upon the corresponding values of $x$. E.g. “the father of $x”$ depends upon $x$ in a way in which “the father of Hannibal” does not depend upon Hannibal: for it not only contains $x$ as a constituent of the meaning, but it varies in a manner dependent upon the variation of $x$. This seems to bring us to the inmost shrine of relatedness. And just as $x$ is no one of its values, so $fx$ is no one of its values.

One difficulty is that we wish to make assertions about $fx$ for all values of $f$, and this is awkward if $f$ can't be detached. But this must simply be allowed.

Consider, in regard to the nature of the variable, the need of any in deductions. We have (say) $(x). \phi x$ and $(x). \phi x \supset \psi x$, and we wish to infer
(x). $\phi x$. This demands that we should replace the premisses by $\phi x$ and $\phi x \ni \psi x$, where the $x$ means simply "any term". The point of this, precisely, is that both $\phi x$ and $\phi x \ni \psi x$ are assertions about one special and particular term, though the term they are about is wholly undefined. And what is more, the $\phi x$ and the $\phi x \ni \psi x$ must both be about the same term. This gives us the variable pure. It is one term, and preserves its identity throughout several propositions, but is not any assignable term. So "the father of x" denotes a single father, but not this or that father. It gives a distributive impartial indication of any one among fathers. The continued identity of the variable is involved in such phrases as "a man who etc.", "if a man does this, he must do that". The who and he refer back to the variable, leaving the value defined solely as the same value as before.

For the sake of definiteness, consider $p \ni q$. This is one value of $\phi(p, q)$; it is in some sense a definite constant object, a value of a variable $\phi(p, q)$. It is the same object as $x \ni y$ or $r \ni s$: these are merely other names for the object. But the kind of object it is is peculiar: it is itself a variable, it cannot take all values, but only such values as result from assigning values to $p$ and $q$. It is not itself an object included among all terms, but all its values are terms.