1. Compute the following derivatives:

(a) \[ \frac{d}{dx} \left[ \sqrt{1 + x^3} \right] \]

(b) \[ \frac{d}{dt} \left[ \frac{1 - t^2}{1 + t^4} \right] \]

(c) \[ \frac{d}{dx} \left[ \frac{\sin(x^2)}{\cos^2(x)} \right] \]

(d) \[ \frac{d}{dt} \left[ e^{t^2 \cos(6t - 1)} \right] \]

2. In studying the biodiversity of an ecosystem, one way of measuring the diversity with a single number is the Shannon diversity index, defined as follows. Suppose that the ecosystem has \( n \) different species, numbered 1, 2, \ldots, \( n \). For \( i = 1, 2, \ldots, n \), let \( p_i \) be the fraction of individuals in the ecosystem that are of species \( i \). (So the \( p_i \)'s are a bunch of numbers, each between 0 and 1, that all add up to 1.) Then the Shannon diversity index of the ecosystem is defined to be

\[
H = -p_1 \ln(p_1) - p_2 \ln(p_2) - \cdots - p_n \ln(p_n) \tag{1}
\]

The idea here is that the higher \( H \) is, the more diverse the ecosystem is.

(a) For a reef ecosystem consisting of 12 parrotfish, 24 angelfish, 50 clownfish, and 14 triggerfish, calculate the Shannon diversity index \( H \).

(b) Why does it make sense that all of the terms in formula (1) have a negative sign in front of them?

(c) Now consider an ecosystem consisting of only two species, say mice and voles. Suppose the fraction of individuals that are mice is \( p \), so the fraction of voles must be \( 1 - p \). Then we can think of the Shannon diversity index as a function of a single variable \( p \), and formula (1) simplifies to

\[
H(p) = -p \ln(p) - (1 - p) \ln(1 - p)
\]

For what values of \( p \) is \( H(p) \) undefined, and why? Graph \( H(p) \), and use your graph to determine \( \lim_{p \to 0} H(p) \) and \( \lim_{p \to 1} H(p) \).

Does your answer make sense biologically (i.e., thinking about what \( H \) is supposed to measure)?
(d) Finally, using the same function $H(p)$ as in part (c), compute the derivative $H'(\frac{1}{2})$. Does your answer make sense based on your graph from part (c)?

3. You may recall having learned a model in LS 30A for an object moving back and forth on the end of a spring. (If you want to review this, you can find it on pages 32–34 in the LS 30 textbook.) The differential equations are

$$\begin{align*}
X' &= V \\
V' &= -65X - 2V
\end{align*}$$

where $X$ represents the position of the object, and $V$ represents its velocity. The constant 65 represents the stiffness of the spring, and the constant 2 represents the friction.

(a) (Review from LS 30) The system just described is linear. Write it in matrix form, and compute the eigenvalues of the matrix. What do those eigenvalues tell you about the behavior of this model? (Remember that this is a continuous time model!)

(b) Now suppose

$$X(t) = e^{-t}\cos(8t) \quad \text{and} \quad V(t) = e^{-t}\left(-\cos(8t) - 8\sin(8t)\right)$$

Compute $X'(t)$. How does $X'(t)$ compare to $V(t)$?

Then compute $V'(t)$, and compute $-65X(t) - 2V(t)$. What do you notice?

Conclude that this pair of functions $X(t)$ and $V(t)$ is a solution to the differential equations given above.

(c) Sketch the graph of $X(t)$, say from $t = 0$ to $t = 5$. Does this time series for $X$ fit with your conclusion(s) from part (a)?

(d) What do you notice about the eigenvalues from part (a), and the numbers that appear in the $X(t)$ function?

4. A recent study found that the amount of carbon monoxide in the air in a typical small suburb can be modeled by the function

$$C(p) = \sqrt{0.45p^2 + 23.6}$$

where $p$ is the population of the suburb (in thousands) and $C$ is the concentration of carbon monoxide (in ppm). Woodside, a suburb of San Francisco, has a population that can be modeled by

$$p(t) = 5.4e^{0.03t}$$

where $t$ is measured in years, starting from now. Use the chain rule to find the rate at which the carbon monoxide level in Woodside will be increasing 5 years from now (i.e., at $t = 5$).
5. As a cell grows in size, it must produce more cytoplasm (the substance that fills most of the volume inside the cell) but also more cell wall material. Yeast cells are approximately spherical, so in this problem we will model a yeast cell as a sphere. Then the volume of cytoplasm inside the cell is \( V = \frac{4}{3}\pi r^3 \), and the area of the cell wall (surface area of the sphere) is \( S = 4\pi r^2 \), where \( r \) is the radius of the sphere. Find a relationship between the rate of change of \( S \) \( \left( \frac{dS}{dt} \right) \) and the rate of change of \( V \) \( \left( \frac{dV}{dt} \right) \). What does this tell you about the rate at which the cell must produce new cell wall material, as the cell grows larger?

6. Now consider a rod-shaped bacteria, which we’ll model as a cylinder, of length \( L \) and radius \( r \). In this case, as the bacterium grows, its radius remains the same while its length increases. Just as in the previous exercise, find a relationship between \( \frac{dS}{dt} \) and \( \frac{dV}{dt} \). Once again, what does this tell you about the rate at which the cell must produce new cell wall material, as the cell grows larger?