Math M32T  
Homework 2  
Due Tuesday, October 15, 2019

1. For each of the following functions, find all critical points, and use the first derivative test (a.k.a number line method, or test points method) to determine if each critical point is a local maximum, local minimum, or neither.

(a) $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 - 6x^2 - 15x + 7$
(b) $f: (0, \infty) \to \mathbb{R}$ defined by $f(t) = t + 4t^{-2}$

2. For each of the following functions, find all critical points, and use the second derivative test to determine (when possible) if each critical point is a local maximum or local minimum.

(a) $f: \mathbb{R} \to \mathbb{R}$ defined by $f(t) = 5 + 36t - 12t^2 - t^3$
(b) $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{x}{4 + x^2}$

Each of the problems below is an applied optimization problem. For each one, you must check that the solution you found does in fact give the maximum or minimum value of the function, using either the first derivative test or the second derivative test (your choice).

3. After administering chemotherapy, a tumor with an initial volume of $4 \text{ cm}^3$ has been reduced to having only $1\%$ proliferating cells, and $99\%$ quiescent (non-proliferating) cells. The volume of proliferating cells grows exponentially, with a growth rate of $0.072$, so that

$$P(t) = 0.04e^{0.072t} = \text{volume of proliferating cells.}$$

The volume of quiescent cells decays exponentially, at a rate of $0.13$, so that

$$Q(t) = 3.96e^{-0.13t} = \text{volume of quiescent cells.}$$

(In both of these, $t$ is measured in days, and $P$ and $Q$ are in $\text{cm}^3$. Note that $0.04$ is $1\%$ of $4$, the initial total volume, and $3.96$ is $99\%$ of $4$.) Treatment should be readministered when the total volume (proliferating cells plus quiescent cells) stops decreasing, and starts to increase again, i.e., when it reaches its minimum. At what value of $t$ will this occur, and how large will the tumor be at that point?

4. (This is the same idea as the previous problem, but starting with different information.)

Suppose that the proliferating cells in a tumor grow exponentially, with a doubling time of $14$ days. Treatment is administered to a tumor with a volume of $9 \text{ cm}^3$, after which $98\%$ of the tumor becomes quiescent, and only $2\%$ remains proliferating. The quiescent cells decay exponentially, with a half life (halving time) of $10$ days. Just as before, we’ll find the time at which the treatment should be readministered.
(a) Find the exponential growth function \( P(t) \) for the volume of the proliferating cells. 

(b) Find the exponential decay function \( Q(t) \) for the volume of the quiescent cells. 

(c) Find the time at which the total tumor volume \( P(t) + Q(t) \) reaches a minimum, which is when the treatment should be readministered.

5. For the ruby-throated hummingbird, a clutch of \( C \) eggs will typically have a survival rate (fraction of eggs that survive) of \( e^{-0.4C} \). Find the clutch size \( C \) that maximizes the total number of offspring that survive.

6. Assume that the Pacific sockeye salmon population, \( S \), grows logistically, with a natural per-capita growth rate (i.e., when the population is small) of 12\% per year, and a carrying capacity of 146 million. As a differential equation, we would write this as

\[
S' = 0.12S \left( 1 - \frac{S}{146} \right)
\]

with \( S \) measured in millions of salmon. Since sockeye salmon are an important commercial fish, humans can harvest the salmon at a rate equal to \( S' \) without disrupting the population. (If, without human interference, the population would grow at a rate of \( S' \) salmon per year, and humans harvest the salmon at exactly the same rate, then the population will neither grow nor decline.) This is considered sustainable harvesting. What should the sockeye salmon population be in order to make \( S' \) as large as possible (and therefore maximize the sustainable harvesting rate)? What is the sustainable harvesting rate (i.e., the value of \( S' \)) at this population level?

7. Consider a yeast cell, which we'll model as a sphere. The cell gains energy by absorption of nutrients through its cell wall, so we can assume that the rate at which it gains energy is proportional to the surface area of the cell, with a proportionality constant of 1.7 \( \text{cal} \mu\text{m}^2\text{h}^{-1} \). Since all parts of the cell are using energy, we can assume that the rate at which the cell loses energy is proportional to its volume, with a proportionality constant of 0.48 \( \text{cal} \mu\text{m}^3\text{h}^{-1} \). What should the radius of the cell be in order to maximize its total increase in energy (gain minus loss)?

8. Networks of blood vessels in animals are shaped in such a way as to approximately minimize the total energy needed to pump blood from the heart to the various organs. One simple way to model this is to consider the hydraulic resistance of blood flowing through blood vessels. For blood flowing a distance of \( L \) through an artery of radius \( r \), the resistance is

\[
k \cdot \frac{L}{r^4}
\]

where \( k \) is a proportionality constant.

Consider blood flowing from a point \( A \), initially through a straight artery, but with a smaller blood vessel branching off to reach a tissue located at point \( C \), which is a
distance $d$ downstream and a perpendicular distance of $p$ away from the main artery. (See the left diagram.) Different choices of the branch angle $\theta$ will lead to different locations of the branch point $B$ along the main artery. (See the right diagram.)

(a) Find the length of the smaller artery $BC$. (Your answer will be in terms of $p$ and $\theta$.)

(b) Find the length of the segment of the main artery $AB$. (Your answer will be in terms of $d$, $p$, and $\theta$.)

(c) Suppose the larger main artery has a radius of $r_1$, and the smaller branch has a radius of $r_2$. Using the formula for resistance given above,

$$\text{resistance} = k \cdot \frac{L}{r^4},$$

write down a formula for the total resistance of blood traveling from $A$ to $B$ to $C$. (That is, the resistance in the segment of artery $AB$ plus the resistance in the segment $BC$.)

(d) Find the branch angle $\theta$ that minimizes the total resistance you found in part (c). Remarkably, almost all of the constants in this problem should disappear, and your final answer should be in terms of just the ratio of the two radii: $\frac{r_2}{r_1}$.

(e) Use your answer to part (d) to find the optimal branch angle if the radius of the smaller artery is 80% of the radius of the larger artery. (That is, if $\frac{r_2}{r_1} = 0.80$.)