1 Problem 1

a) The curve $C$ is represented as $z(t) = F(t) + iG(t)$. When the coordinates are transformed such that $z(w) = F(w) + iG(w)$, the curve is mapped to $w = t$, which is of course part of the real line.

b) A natural parameterization of the ellipse is given by

\begin{align*}
    x &= a \cos t \\
    y &= b \sin t.
\end{align*}

Therefore, a transformation which maps the ellipse to the real line is given by $z(w) = a \cos w + ib \sin w$.

2 Problem 2

a) This boundary value problem becomes straightforward in the right coordinates. Using $\theta \in [0, 2\pi)$ instead of $(x, y)$, it is immediately clear that a solution is given by $f(\theta) = a\theta + b$. Note that this solution is in fact harmonic, since it happens to be the imaginary part of $C_1 \log z + C_2$. The boundary conditions are then $f(0) = 1$, $f(\pi) = -1$. The solution is thus

\begin{align*}
    f(\theta) &= -\frac{2}{\pi} \theta + 1 \\
    f(x, y) &= -\frac{2}{\pi} \arctan \frac{y}{x} + 1.
\end{align*}

b) A solution to this problem may be found by adding together solutions from part (a) centered at $x = -1$ and $x = 1$.

\begin{align*}
    f(x, y) &= A \left[ -\frac{2}{\pi} \arctan \frac{y}{x+1} + 1 \right] + B \left[ -\frac{2}{\pi} \arctan \frac{y}{x-1} + 1 \right]
\end{align*}

Applying the boundary conditions, we find the solution

\begin{align*}
    f(x, y) &= -\frac{1}{\pi} \arctan \frac{y}{x+1} - \frac{1}{\pi} \arctan \frac{y}{x-1} + 1.
\end{align*}

3 Problem 3

Take the wires to have linear charge densities $\lambda$ and $-\lambda$, and the wires to run perpendicular to the complex plane, so that the wires exist in $\mathbb{R} \times \mathbb{C}$. Then taking a two-dimensional cross section of the wires, the holomorphic potential will be

\begin{equation}
    \Omega(z) = \lambda \log \left( \frac{z - z_-}{z - z_+} \right),
\end{equation}

where the positive and negative charges are situated at $z_+$ and $z_-$, respectively. The electrostatic potential will then be the real part of $\Omega$.

\begin{equation}
    \Phi(x, y) = \frac{\lambda}{2} \log \left( \frac{(x - x_-)^2 + (y - y_-)^2}{(x - x_+)^2 + (y - y_+)^2} \right)
\end{equation}
4 Problem 4

To solve this problem, we will first consider the capacitance per unit length of a more basic system, with concentric cylindrical conductors, then apply a conformal transformation so as to obtain the correct equation for this particular situation. In the case of coaxial cylinders, the potential takes the form of the real part of the holomorphic potential.

\[ \Phi (\rho) = a \log \rho + b \]  

(4.1)

The boundary conditions are obtained by setting the potential at the inner radius \( \rho_1 \) to \( V_1 \) and the potential at the outer radius \( \rho_2 \) to \( V_2 \). The potential is then

\[ \Phi (\rho) = \frac{V_2 - V_1}{\log \frac{\rho_2}{\rho_1}} \log \rho + \frac{V_1 \log \rho_2 - V_2 \log \rho_1}{\log \frac{\rho_2}{\rho_1}}. \]  

(4.2)

To find the capacitance, we must first obtain the charge in a cross-sectional slice of the cylinder, using \( \nabla^2 \Phi = 4\pi \rho \).

\[ q = \frac{1}{4\pi} \int_{\rho_1}^{\rho_2} d\rho \int_0^{2\pi} \rho d\phi \frac{1}{\rho} \partial_{\rho} \left( \rho \partial_{\rho} \Phi \right) \]  

(4.3)

\[ = \frac{V_2 - V_1}{2 \log \frac{\rho_2}{\rho_1}}. \]  

(4.4)

The capacitance per unit length, \( C = \frac{q}{V} \), is therefore

\[ C = \frac{1}{2 \log \frac{\rho_2}{\rho_1}}. \]  

(4.5)

Now that we have an expression for the capacitance in the cylindrical case, we must find a transformation that maps this configuration to the one in the problem. The map between ellipses (\( z \)) and circles (\( w \)) is given by

\[ z = \frac{c}{2} \left( w - \frac{1}{w} \right), \]  

(4.6)

where the ellipses are given by \( |z - c| + |z + c| = R \). This implies that an ellipse is mapped to a circle with radius

\[ \rho = \frac{1}{2c} \left( R + \sqrt{R^2 - 4c^2} \right). \]  

(4.7)

Therefore the capacitance per unit length is

\[ C = \frac{1}{2 \log \frac{R_2 + \sqrt{R_2^2 - 4c^2}}{R_1 + \sqrt{R_1^2 - 4c^2}}}. \]  

(4.8)