1. Let $f(x, y) = x^2 - y^2$. Compute the directional derivative $D_{\vec{u}} f(1, 3)$ in the direction $\vec{u} = \begin{bmatrix} 4 \\ 5 \\ -3 \\ 5 \end{bmatrix}$.

2. Let $f(x, y, z) = x^3 - xy + yz^2 - 3xz$. Compute the directional derivative of $f$ at the point $(2, 5, -1)$, in the direction parallel to the vector $\begin{bmatrix} -3 \\ 6 \\ -6 \end{bmatrix}$.

3. The temperature at each point in $xyz$-space is given by

   $T(x, y, z) = 3x^2 + y^3z - xyz$

   A bug is currently at the point $(2, 1, 3)$, and it plans to fly in a straight line to the point $(-1, -1, 9)$. (Assume $x$, $y$, and $z$ are measured in meters, and $T$ is in Celsius.)

   (a) What is the average rate of change of the temperature (per meter) between the points $(2, 1, 3)$ and $(-1, -1, 9)$?

   (b) What is the rate of change of the temperature (per meter) at the instant that the bug sets out from the point $(2, 1, 3)$? (Hint: If the bug goes in a straight line from $(2, 1, 3)$ to $(-1, -1, 9)$, what vector does it follow?)

4. You are walking on a landscape that can be described as the graph of

   $f(x, y) = 6 - x^4 - y^2 + 4xy - 3y$

   You are currently at the point $(1, 2)$.

   (a) Suppose you like a challenge, so you want to go uphill in the steepest way possible. In what direction should you walk? If you go in that direction, what will the slope of your path be?

   (b) Water flows downhill following the steepest path possible. If it starts raining, in what direction will the water flow at the point where you’re standing? What is the slope of the hill in that direction?

   (c) Suppose you’re too tired to walk uphill, and walking downhill hurts your knees. Which two directions could you walk in so that you would neither go uphill nor downhill at all?

5. A cat is sleeping on a hot tin roof. We’ll consider the southwest corner of the roof to be the origin, $(0, 0)$, with the positive $x$ axis pointing east and the positive $y$ axis pointing
north (both measured in meters). With this coordinate system, the temperature at any point \((x, y)\) on the roof is given by

\[
T(x, y) = 20 + 2x + xy + y^2 \quad \text{(in degrees C)}
\]

The cat’s current location is \((2, 1)\). He wakes up and feels too cold, so he decides to move.

(a) If the cat moves due east, at what rate (in degrees C per meter) would the temperature increase/decrease?

(b) If the cat moves due northeast, at what rate would the temperature increase/decrease?

(c) In what direction should the cat go in order for the temperature to increase the fastest? If he goes in that direction, at what rate would the temperature increase?

6. The Anna’s hummingbird is a common sight on the UCLA campus. Like all hummingbirds, they have the remarkable ability to fly omnidirectionally and to hover in mid-air. They feed primarily on nectar from flowers, which they find with the help of their sense of smell. Suppose an Anna’s hummingbird is hovering in the air, trying to sniff its way to a nearby flower. The bird’s current position is \((4, 7, 5)\), and the density of the flower’s odorants in the air (i.e., how strong the smell is at any point \((x, y, z)\)) is given by

\[
S(x, y, z) = e^{-x^2-y^2-(z-1)^2}
\]

If the hummingbird always follows his nose, that is, flies in the direction in which the smell increases the fastest, in what direction will he fly?

7. A fitness landscape for two traits \(x\) and \(y\) is modeled by the function

\[
f(x, y) = 10 - x^2 + 2x + 3xy - y^4
\]

This function has a single local maximum (which is also its global maximum), and it is somewhere near \(x = 1, y = -2\). You know that “following the gradient vector field” will lead you to a maximum. So use Euler’s method, with the gradient vector field, starting from the initial point \((1, -2)\), to get closer to the maximum. Do three steps of Euler’s method, with a step size of 0.1.

(If you want, you can code this in SageMath or some other system, and run it for many many steps to get a really close approximation to the maximum point.)