1 Problem 1

a) Consider any two points on $S^2$. Clearly there is always a path that connects them. Now consider how one would disconnect the two points. It would require removing a circle (or something homeomorphic to a circle) around one of the points. Thus, removing any finite number of points will leave the space path connected, and therefore connected.

b) As we will explain in part (e), $V_1$ is homeomorphic to $\mathbb{R}^2$. Since homeomorphisms don’t affect compactness, this means that $V_1$ is non-compact, as are all $V_n$ for $n > 1$. $V_0$, however, is just $S^2$, which is a compact space.

c) Again, since homeomorphisms preserve path-connectedness, it is clear (given part (e)) that $V_0$ and $V_1$ are path-connected, while $V_n$ for $n > 1$ is not.

d) The fundamental group of $V_n$ is $\mathbb{Z}^{n-1}$ (for $n > 1$). This is because a loop around the $n^{th}$ removed point may be constructed from loops around the first $n + 1$ removed points.

e) Placing the removed point at the north pole, the stereographic projection may be used to map $V_1$ to $\mathbb{R}^2$. Similarly, moving the two removed points of $V_2$ at the north and south poles, the Mercator projection may be used as a homeomorphism between $V_2$ and the infinite cylinder.

2 Problem 2

First of all making the transformation recommended, the integral becomes

$$I = \oint_C \frac{dz}{iz \left[a + \frac{1}{2} (z + \bar{z}) + \frac{1}{2\pi i} (z - \bar{z}) \right]}$$

$$= -2i \oint_C \frac{dz}{(b - ic) z^2 + 2az + b + ic},$$

where $C$ is counterclockwise around the unit circle. This integrand has two poles, one of which lies within the unit circle (since $|z_+| |z_-| = 1$).

$$z_\pm = \frac{-a \pm \sqrt{a^2 - b^2 - c^2}}{b - ic}$$

Assuming that $a > 0$, the integral is found by taking the residue at $z_+$.  

$$I = -2\pi i \times \frac{2i}{b - ic} \times \frac{b - ic}{2\sqrt{a^2 - b^2 - c^2}}$$

$$= \frac{2\pi}{\sqrt{a^2 - b^2 - c^2}}$$

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3 Problem 3

a) The integral $A$ may be written as

$$A = \frac{1}{2} \int_{-\infty}^{\infty} dx \frac{\cos (kx)}{x^2 + m^2}$$

$$= \frac{1}{2} \text{Re} \left[ \int_{-\infty}^{\infty} dx \frac{e^{ikx}}{x^2 + m^2} \right].$$  \hspace{1cm} (3.1)

Now, since the integrand in the upper half of the complex plane when $|z| \to \infty$ is zero, the integral may be equivalently expressed as

$$A = \frac{1}{2} \text{Re} \left[ \oint_{C} dz \frac{e^{ikz}}{z^2 + m^2} \right].$$  \hspace{1cm} (3.2)

where $C = \mathcal{L} \cup S_R$ and $\mathcal{L} = [-R, R]$ along the real line and $S_R$ is the semicircle of radius $R$ in the upper half of the complex plane, both taken in the limit $R \to \infty$. Since the integrand has one pole within the integration loop, the integral becomes

$$A = \frac{1}{2} \text{Re} \left[ 2\pi i \left( \frac{e^{-km}}{2im} \right) \right]$$

$$= \frac{\pi}{2m} e^{-km}.$$  \hspace{1cm} (3.3)

b) Similar to what we did in part (a), we first transform the integral to be

$$B = \frac{1}{2} \text{Im} \left[ \int_{-\infty}^{\infty} dz \frac{e^{iz}}{z} \right].$$  \hspace{1cm} (3.4)

Now consider the related integral

$$\frac{1}{2} \int_{C} dz \frac{e^{iz}}{z}.$$  \hspace{1cm} (3.5)

If we take our contour to be the same as in part (a), we run into trouble with a pole on the real axis at $z = 0$. Therefore, we must take the contour to avoid this point. A convenient choice is $C = \mathcal{L}_1 \cup S_\epsilon \cup \mathcal{L}_2 \cup S_R$, where $\mathcal{L}_1 = [-R, -\epsilon]$, $\mathcal{L}_2 = [\epsilon, R]$, and $S_R$ and $S_\epsilon$ are semicircles centered at the origin with radius $R$ and $\epsilon$, respectively. Note that $S_\epsilon$ is traversed clockwise while $S_R$ is traversed counterclockwise. In the end, we will take the limits $\epsilon \to 0$ and $R \to \infty$. Of course, this total integral is zero, since no poles are enclosed. Then noticing that the integrand vanishes as $|z| \to \infty$,

$$0 = \int_{\mathcal{L}_1 \cup \mathcal{L}_2} dz \frac{e^{iz}}{z} + \lim_{\epsilon \to 0} \int_{S_\epsilon} dz \frac{e^{iz}}{z}.$$  \hspace{1cm} (3.6)

Since $S_\epsilon$ is just a half semicircle (oriented clockwise) around the pole at the origin, the integral is simply $-i\pi$. Therefore, the original integral is

$$B = -\frac{1}{2} \text{Im} \left[ \lim_{\epsilon \to 0} \int_{S_\epsilon} dz \frac{e^{iz}}{z} \right]$$

$$= \frac{\pi}{2}.$$  \hspace{1cm} (3.7)
c) Clearly there is only one pole, at $z = 0$ which happens to be an essential singularity. To find the residue, we will use the Taylor expansion of the integrand.

$$e^{1/z} \sin 1/z = \left(1 + \frac{1}{z} + \frac{1}{2z^2} + \ldots\right) \left(\frac{1}{z} - \frac{1}{6z^3} + \ldots\right)$$  \hfill (3.12)

$$= \frac{1}{z} + \frac{1}{z^2} + \ldots$$  \hfill (3.13)

The residue is just the coefficient of the $\frac{1}{z}$ term in the Laurent expansion, which in this case is 1. Therefore, the integral is $C = 2\pi i$.

4 Problem 4

a) Note first that the integrand has poles at all $n \in \mathbb{Z}$. The contour may be continuously deformed to encircle all the poles on the real axis. Thus we should take the contours to run clockwise so as to cancel all contributions except for the contours encircling the poles and the contour at $|z| = R$. As $R \to \infty$, this contribution is killed, so we are left with only the contours encircling each integer on the real line. Therefore, the integral is

$$\oint_C dz \frac{f(z)}{e^{2\pi iz} - 1} = 2\pi i \sum_{n \in \mathbb{Z}} \text{Res}_{z=n} \left[ \frac{f(z)}{e^{2\pi iz} - 1} \right]$$  \hfill (4.1)

$$= 2\pi i \sum_{n \in \mathbb{Z}} \lim_{z \to n} \frac{(z - n) f(z)}{e^{2\pi iz} - 1}$$  \hfill (4.2)

$$= 2\pi i \sum_{n \in \mathbb{Z}} \lim_{z \to n} \frac{f(z) + (z - n) f'(z)}{2\pi i e^{2\pi iz}}$$  \hfill (4.3)

$$= \sum_{n \in \mathbb{Z}} f(n).$$  \hfill (4.4)

b) Using the result of part (a), but now considering taking the contour in the opposite (counterclockwise) direction,

$$\sum_{n \in \mathbb{Z}} \frac{1}{(n + a)^2} = -2\pi i \text{Res}_{z=-a} \left[ \frac{1}{e^{2\pi iz} - 1} \right]$$  \hfill (4.5)

$$= -2\pi i \lim_{z \to -a} \frac{d}{dz} \left[ \frac{1}{e^{2\pi iz} - 1} \right]$$  \hfill (4.6)

$$= -4\pi^2 \frac{\partial}{\partial a} \frac{1}{e^{2\pi ia} - 1}$$  \hfill (4.7)

$$= \frac{\pi^2}{2} \csc^2(\pi a).$$  \hfill (4.8)