Midterm 1

Last Name: ___________________________
First Name: ___________________________
Student ID: ___________________________

By signing below, you affirm that you have neither given nor received unauthorized help on this exam.

Signature: ___________________________

Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above. You may not use books, notes, or any other material to help you. You may use a calculator, but not a programmable or graphing calculator. Please make sure your phone is silenced and stowed out of sight. You may use any available space on the exam for scratch work, including the backs of the pages. If you need more scratch paper, please ask one of the proctors.

Please do not write below this line.

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1. The flow rate, $B$, in $\frac{\text{mL}}{\text{s}}$, of blood through a blood vessel is given by

$$B = \frac{1}{200} R^4 (P - 40)$$

where $R$ is the radius of the blood vessel (in mm), $P$ is the pressure at the start of the blood vessel (in mmHg).

In a patient with atherosclerosis, buildup of plaque inside an artery is decreasing the artery’s radius $R$, and decreasing the flow rate $B$ of blood through the artery. This leads to a change in the blood pressure, $P$.

(a) (7 points) Find a formula that relates the rates of change of $B$, $P$, and $R$ to each other.

(b) (4 points) Suppose you have measured the following about the left coronary artery of a patient with atherosclerosis:

radius: currently 2 mm, decreasing at 0.3 mm per month
blood pressure: currently 110 mmHg
blood flow rate: increasing at 0.1 $\frac{\text{mL}}{\text{s}}$ per month

How fast is the patient’s blood pressure changing, and is it increasing or decreasing?

Question 1 continues on the next page...
Question 1 continued...
2. (10 points) A robin will forage for food no farther than a certain distance $r$ from its nest. So the area over which it forages is a circle, with its nest at the center, of radius $r$. The larger the area, of course, the more food it will find, so we can assume that the amount of energy that the robin consumes is proportional to the area of this circle, with proportionality constant $6 \frac{\text{kcal}}{\text{km}^2}$.

On the other hand, the larger its foraging territory is, the more energy the robin must expend defending this territory from other birds. Researchers have found that this energy is proportional to the radius cubed ($r^3$), with a proportionality constant $2\pi \frac{\text{kcal}}{\text{km}^3}$.

Assume the robin has evolved to instinctively choose a radius $r$ that maximizes its net energy gain:

$$\text{net gain} = \left( \text{energy gained from foraging for food} \right) - \left( \text{energy spent defending territory} \right)$$

Find this optimal radius. Be sure to check that what you’ve found gives the maximum energy gain.
Question 2 continued...
3. Consider the line passing through two points \( A = (6, 13, 1) \) and \( B = (1, 8, 1) \). In this problem we will compute the distance from the point \( P = (7, 6, 8) \) to this line.

(a) (2 points) Start by computing the vector \( \vec{v} \) from \( A \) to \( P \), and the vector \( \vec{w} \) from \( A \) to \( B \).

(b) (6 points) Compute the vector \( \vec{v}_\parallel \) shown in the picture below.
(c) (4 points) Compute the vector $\vec{v}_\perp$ shown in the picture below*. Finally, use this to find the distance from $P$ to the line.

*Hint: What is the relationship between $\vec{v}_\parallel$, $\vec{v}_\perp$, and $\vec{v}$?
4. (12 points) Peregrine falcons prey primarily on other birds. They are known for their incredible speed, and in particular for catching birds in mid-air by diving onto them at speeds of over 200 mph. Suppose a falcon starts a dive at the point \((1, -3, 90)\), and dives in a straight line to intercept its prey at the point \((-2, 3, 68)\). The air pressure at any point \((x, y, z)\) is given by

\[
P(x, y, z) = \frac{10 - 5ye^{-\frac{1}{10}(x^2 + y^2)}}{10 + z}
\]

Note: The whole expression is \(-\frac{1}{10}(x^2 + y^2)\) is in the exponent of \(e\) here. Assume the \(x, y, z\)-coordinates are in meters, and the pressure \(P\) is in atm.

What is the (instantaneous) rate of change of the air pressure (per meter) at the instant that the falcon starts its dive?

\((Hint: \text{ It will help to first find the vector from } (1, -3, 90) \text{ to } (-2, 3, 68).)\)
Question 4 continued...
Some useful formulas, etc:

Differentiation rules:

\[ \frac{d}{dx}[e^x] = e^x \]

\[ \frac{d}{dx}[\ln(x)] = \frac{1}{x} \]

\[ \frac{d}{dx}[/text{sin}(x)] = \cos(x) \]

\[ \frac{d}{dx}[/text{cos}(x)] = -\sin(x) \]

\[ \frac{d}{dx}[x^n] = nx^{n-1} \quad \text{(Power Rule)} \]

\[ \frac{d}{dx}[f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x) \quad \text{(Product Rule)} \]

\[ \frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g(x)^2} \quad \text{(Quotient Rule)} \]

\[ \frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x) \quad \text{(Chain Rule)} \]