1. The flow rate, \( B \), in \( \frac{\text{ml}}{\text{s}} \), of blood through a blood vessel is given by

\[
B = \frac{1}{200} R^4 (P - 40)
\]

where \( R \) is the radius of the blood vessel (in mm), \( P \) is the pressure at the start of the blood vessel (in mmHg).

In a patient with atherosclerosis, buildup of plaque inside an artery is decreasing the artery’s radius \( R \), and decreasing the flow rate \( B \) of blood through the artery. This leads to a change in the blood pressure, \( P \).

(a) (7 points) Find a formula that relates the rates of change of \( B \), \( P \), and \( R \) to each other.

\[
\frac{dB}{dt} = \left( \frac{1}{200} R^3 \right) \frac{dR}{dt} (P - 40) + \left( \frac{1}{200} R^4 \right) \frac{dP}{dt} - 0 \quad \text{Product Rule!}
\]

\[
\frac{dB}{dt} = \frac{1}{50} R^3 \frac{dR}{dt} (P - 40) + \frac{1}{200} R^4 \frac{dP}{dt}
\]

(b) (4 points) Suppose you have measured the following about the left coronary artery of a patient with atherosclerosis:

- radius: currently 2 mm, decreasing at 0.3 mm per month
- blood pressure: currently 110 mmHg
- blood flow rate: increasing at 0.1 \( \frac{\text{ml}}{\text{s}} \) per month

How fast is the patient’s blood pressure changing, and is it increasing or decreasing?

This means:

\[
R = 2 \text{ mm}, \quad \frac{dR}{dt} = -0.3 \text{ mm/month}
\]

\[
P = 110 \text{ mmHg}
\]

\[
\frac{dB}{dt} = 0.1 \text{ ml/s per month}
\]

Find: \( \frac{dP}{dt} \)

Question 1 continues on the next page...
Question 1 continued...

\[ 0.1 = \frac{1}{50} \cdot 2^3 \cdot (-0.3) \cdot (110 - 40) + \frac{1}{200} \cdot 2^9 \cdot \frac{dp}{dt} \]

\[ 0.1 = -3.36 + 0.08 \frac{dp}{dt} \]

\[ 0.08 \frac{dp}{dt} = 0.1 + 3.36 = 3.46 \]

\[ \frac{dp}{dt} = \frac{3.46}{0.08} = 43.25 \text{ \text{month}} \]

Blood pressure is increasing at 43.25 \text{ month/month}.

This is dangerous... patient will probably be dead within two months.
2. (10 points) A robin will forage for food no farther than a certain distance \( r \) from its nest. So the area over which it forages is a circle, with its nest at the center, of radius \( r \). The larger the area, of course, the more food it will find, so we can assume that the amount of energy that the robin consumes is proportional to the area of this circle, with proportionality constant \( 6 \text{ kcal/km}^2 \).

On the other hand, the larger its foraging territory is, the more energy the robin must expend defending this territory from other birds. Researchers have found that this energy is proportional to the radius cubed \( (r^3) \), with a proportionality constant \( 2\pi \text{ kcal/km}^3 \).

Assume the robin has evolved to instinctively choose a radius \( r \) that maximizes its net energy gain:

\[
\text{net gain} = \left( \text{energy gained from foraging for food} \right) - \left( \text{energy spent defending territory} \right)
\]

Find this optimal radius. Be sure to check that what you’ve found gives the maximum energy gain.

\[
\begin{align*}
\text{Energy gained} & = \left( 6 \text{ kcal/km}^2 \right) \cdot (\pi r^2) = 6\pi r^2 \\
\text{Energy spent} & = \left( 2\pi \text{ kcal/km}^3 \right) \cdot (r^3) = 2\pi r^3 \\
\text{Net energy gain} & = 6\pi r^2 - 2\pi r^3 \\
\text{Call this} \ E(r) & = 6\pi r^2 - 2\pi r^3, \quad \leftarrow \text{Maximize this.}
\end{align*}
\]

\[
\begin{align*}
E'(r) & = 12\pi r - 6\pi r^2 \\
E'(r) \text{ undefined: Nowhere} \\
E'(r) = 0 & : \quad 12\pi r - 6\pi r^2 = 0 \\
& \quad 6\pi \cdot r \cdot (2-r) = 0 \\
& \quad r=0 \quad \text{ or } \quad r=2 \quad \leftarrow \text{Two critical points}
\end{align*}
\]

Question 2 continues on the next page...
Question 2 continued...

Also: Boundary point at \( r = 0 \). \( E(0) = 0 \), so this should not be a max.

Test critical points:

One way: First derivative test (number line)

\[
\begin{array}{c}
E'(r): \quad & \text{positive} & \quad & \text{negative} \\
0 \quad & \quad & \quad & 2 \\
E \quad & \text{increasing} & \quad & \text{decreasing}
\end{array}
\]

So the maximum net gain of energy is at \( r = 2 \) km.

Other way: Second derivative test

\[
E''(r) = 12\pi - 12\pi r = 12\pi(1-r)
\]

\( E''(0) = 12\pi > 0 \), so there's a local minimum at \( 0 \).

\( E''(2) = 12\pi(-1) = -12\pi < 0 \), so there's a local maximum at \( r = 2 \).

Again we see that the maximum energy gain is at \( r = 2 \) km.

So the robin should evolve to forage over a radius of 2 km from its nest.
3. Consider the line passing through two points $A = (6, 13, 1)$ and $B = (1, 8, 1)$. In this problem we will compute the distance from the point $P = (7, 6, 8)$ to this line.

(a) (2 points) Start by computing the vector $\vec{v}$ from $A$ to $P$, and the vector $\vec{w}$ from $A$ to $B$.

$$\vec{v} = \begin{bmatrix} 7 - 6 \\ 6 - 13 \\ 8 - 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -7 \\ 7 \end{bmatrix}$$

$$\vec{w} = \begin{bmatrix} 1 - 6 \\ 8 - 13 \\ 1 - 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -5 \\ 0 \end{bmatrix}$$

(b) (6 points) Compute the vector $\vec{v}_\parallel$ shown in the picture below.

$$\vec{v}_\parallel = \text{proj}_w(\vec{v}) = \left( \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2} \right) \vec{w} = (\vec{v} \cdot \vec{w}) \frac{\vec{w}}{\|\vec{w}\|^2}$$

$$= \left( \frac{-5 + 35 + 0}{25 + 25 + 0} \right) \begin{bmatrix} -5 \\ -5 \\ 0 \end{bmatrix} = \left( \frac{30}{50} \right) \begin{bmatrix} -5 \\ -5 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ 0 \end{bmatrix}$$

Question 3 continues on the next page...
(c) (4 points) Compute the vector $\vec{v}_\perp$ shown in the picture below. Finally, use this to find the distance from $P$ to the line.

Since $\vec{v}_\parallel + \vec{v}_\perp = \vec{v}$,

$$\vec{v}_\perp = \vec{V} - \vec{v}_\parallel = \begin{bmatrix} 1 \\ -7 \\ 7 \end{bmatrix} - \begin{bmatrix} -3 \\ -3 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -4 \\ 7 \end{bmatrix}$$

Distance from $P$ to line $= ||\vec{v}_\perp|| = \sqrt{4^2 + (-4)^2 + 7^2}$

$$= \sqrt{81} = 9$$

*Hint: What is the relationship between $\vec{v}_\parallel$, $\vec{v}_\perp$, and $\vec{v}$?
4. (12 points) Peregrine falcons prey primarily on other birds. They are known for their incredible speed, and in particular for catching birds in mid-air by diving onto them at speeds of over 200 mph. Suppose a falcon starts a dive at the point \((1, -3, 90)\), and dives in a straight line to intercept its prey at the point \((-2, 3, 68)\). The air pressure at any point \((x, y, z)\) is given by

\[
P(x, y, z) = \frac{10 - 5ye^{-\frac{1}{10}(x^2+y^2)}}{10 + z} = \left(\frac{1}{10+z}\right) \cdot (10 - 5ye^{-\frac{1}{10}(x^2+y^2)})
\]

Note: The whole expression is \(-\frac{1}{10}(x^2 + y^2)\) is in the exponent of e here. Assume the \(x, y, z\)-coordinates are in meters, and the pressure \(P\) is in atm.

What is the (instantaneous) rate of change of the air pressure (per meter) at the instant that the falcon starts its dive?

\((\text{Hint: It will help to first find the vector from } (1, -3, 90) \text{ to } (-2, 3, 68)).\)

\[\begin{align*}
\text{Vector:} \quad \vec{V} &= \begin{bmatrix} -2 & -1 \\ 3 & -3 \\ 68 & -90 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \\ -22 \end{bmatrix} \\
\|\vec{V}\| &= \sqrt{(-3)^2 + 6^2 + (-22)^2} = \sqrt{529} = 23
\end{align*}\]

\[\text{Unit vector:} \quad \hat{\vec{V}} = \frac{\vec{V}}{\|\vec{V}\|} = \frac{1}{23} \begin{bmatrix} -3 \\ 6 \\ -22 \end{bmatrix} = \begin{bmatrix} -3/23 \\ 6/23 \\ -22/23 \end{bmatrix}\]

\[\begin{align*}
\text{Gradient:} \quad \frac{\partial P}{\partial x} &= \frac{1}{10+z} \cdot \left(-5ye^{-\frac{1}{10}(x^2+y^2)} \cdot \frac{1}{10}(2x)\right) \\
&= \frac{1}{10+z} \cdot y e^{-\frac{1}{10}(x^2+y^2)} \left(-\frac{1}{10} \cdot 2x\right) \\
&= \frac{x}{10+z} \cdot (-3) e^{-1} = \frac{-3e^{-1}}{100} = -0.0110
\end{align*}\]

Question 4 continues on the next page...
Question 4 continued...

\[ \frac{dp}{dy} = \frac{1}{10 + z} \left[ (-5) \cdot e^{-\frac{4}{10} (y^2 + x^2)} + (5y) \cdot e^{-\frac{4}{10} (y^2 + x^2)} \cdot (-2 \cdot 2y) \right] \]

\[ = \frac{1}{10 + z} \cdot e^{-\frac{4}{10} (y^2 + x^2)} \cdot ( -5 + y^2 ) \]

\[ \left. \frac{dp}{dy} \right|_{(1, -3, 10)} = \frac{1}{10 + 90} \cdot e^{-1} \cdot (-5 + 9) = \frac{4 \cdot e^{-1}}{100} = 0.0147 \]

\[ \frac{dp}{dz} = - (10 + z)^{-2} \cdot (1) \cdot (10 - 5ye^{-\frac{4}{10} (y^2 + x^2)}) = - \frac{10 - 5ye^{-\frac{4}{10} (y^2 + x^2)}}{(10 + z)^2} \]

\[ \left. \frac{dp}{dz} \right|_{(1, -3, 10)} = - \frac{10 - 5 \cdot -3 \cdot e^{-1}}{(10 + 90)^2} = -0.00155 \]

So, \[ \nabla p = \begin{bmatrix} \frac{1}{10 + z} \cdot xe^{-\frac{4}{10} (y^2 + x^2)} \\ \frac{1}{10 + z} \cdot ye^{-\frac{4}{10} (y^2 + x^2)} \\ \frac{-1}{(10 + z)^2} \cdot (10 - 5ye^{-\frac{4}{10} (y^2 + x^2)}) \end{bmatrix} \]

and \[ \left. \nabla p \right|_{(1, -3, 10)} = \begin{bmatrix} -0.0110 \\ 0.0147 \\ -0.00155 \end{bmatrix} \]

Rate of change of pressure = \[ D_a p = \nabla p \cdot \mathbf{u} \]

\[ = \begin{bmatrix} -0.0110 & 0.0147 & -0.00155 \end{bmatrix} \cdot \begin{bmatrix} -\frac{3}{2} \cdot 3 \\ \frac{3}{2} \cdot 3 \\ -\frac{2}{2} \cdot 3 \end{bmatrix} = \begin{bmatrix} 0.00676 \text{ atm/m} \end{bmatrix} \]