1 Problem 1

Integrating both sides against a smooth function with compact support, as is suggested,

\[
\int_{-\infty}^{\infty} dx f(x) \lim_{\epsilon \to 0} \left( \frac{1}{x - y - i\epsilon} - \frac{1}{x - y + i\epsilon} \right) = 2\pi i f(y). \tag{1.1}
\]

In order to prove that this is true, complexify the integral and choose the contours to be semicircles surrounding the upper half and lower half planes. Since \( f \) is well-behaved, the integrand becomes zero as \( |z| \to \infty \). The poles encircled by the contours are those at \( z = y \pm i\epsilon \). Notice that the upper contour runs counterclockwise and the lower contour runs clockwise.

\[
2 \int_{-\infty}^{\infty} dx f(x) \lim_{\epsilon \to 0} \left( \frac{1}{x - y - i\epsilon} - \frac{1}{x - y + i\epsilon} \right) = \oint_{C_1 \cup C_2} dz f(z) \lim_{\epsilon \to 0} \left( \frac{1}{z - y - i\epsilon} - \frac{1}{z - y + i\epsilon} \right) \tag{1.2}
\]

\[
= 2\pi i \lim_{\epsilon \to 0} \{ \text{Res}_{z=y+i\epsilon} [f(z)] + \text{Res}_{z=y-i\epsilon} [f(z)] \} \tag{1.3}
\]

\[
= 4\pi i f(y) \tag{1.4}
\]

This implies that

\[
\int_{-\infty}^{\infty} dx f(x) \lim_{\epsilon \to 0} \left( \frac{1}{x - y - i\epsilon} - \frac{1}{x - y + i\epsilon} \right) = 2\pi i f(y), \tag{1.5}
\]

as we set out to prove. We may also approach this problem without using complex analysis. Using the fact that

\[
\frac{1}{x - y - i\epsilon} - \frac{1}{x - y + i\epsilon} = \frac{2i\epsilon}{(x - y)^2 + \epsilon^2} \tag{1.7}
\]

and integrating against a function \( f(x) \) of compact support and changing variables so that \( u = \frac{x - y}{\epsilon} \),

\[
\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{2i\epsilon f(x) dx}{(x - y)^2 + \epsilon^2} = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{2i\epsilon^2 f(y + \epsilon u) du}{\epsilon^2 u^2 + \epsilon^2} \tag{1.8}
\]

\[
= \int_{-\infty}^{\infty} \frac{2if(y) du}{u^2 + 1} \tag{1.9}
\]

\[
= 2\pi i f(y), \tag{1.10}
\]

as we set out to prove.

2 Problem 2

First we rewrite the sum on the left hand side as
\[
\lim_{\epsilon \to 0} \sum_{n \in \mathbb{Z}} e^{2\pi i n (x-y) - 2\pi |n| \epsilon} = \lim_{\epsilon \to 0} \sum_{n=0}^{\infty} \left[ e^{2\pi i n (x-y+\epsilon)} + e^{2\pi i n (x-y-\epsilon)} \right] - 1
\] (2.1)

\[
= \lim_{\epsilon \to 0} \left[ \frac{1}{1 - e^{2\pi i (x-y+\epsilon)}} + \frac{1}{1 - e^{-2\pi i (x-y-\epsilon)}} - 1 \right]
\] (2.2)

Integrating both sides as in part (a) against a smooth function \( f \) with compact support (and using the same contour),

\[
2 \int_{-\infty}^{\infty} dx f(x) \lim_{\epsilon \to 0} \sum_{n \in \mathbb{Z}} e^{2\pi i n (x-y) - 2\pi |n| \epsilon}
\] (2.3)

\[
= \lim_{\epsilon \to 0} 2\pi i \sum_{m \in \mathbb{Z}} \left\{ -\text{Res}_{z=m+y-i\epsilon} \left[ \frac{f(z)}{1 - e^{2\pi i (z-y+i\epsilon)}} \right] + \text{Res}_{z=m+y+i\epsilon} \left[ \frac{f(z)}{1 - e^{-2\pi i (z-y-i\epsilon)}} \right] \right\}
\] (2.4)

\[
= \lim_{\epsilon \to 0} 2\pi i \sum_{m \in \mathbb{Z}} \left[ \frac{f(m+y-i\epsilon)}{2\pi i} + \frac{f(m+y+i\epsilon)}{2\pi i} \right]
\] (2.5)

\[
= 2 \sum_{m \in \mathbb{Z}} f(m+y).
\] (2.6)

Notice that the pole below the axis picks up a negative sign because we encircle it clockwise. As in the previous problem, this equality implies what we set out to prove, that

\[
\lim_{\epsilon \to 0} \sum_{n \in \mathbb{Z}} e^{2\pi i n (x-y) - 2\pi |n| \epsilon} = \sum_{m \in \mathbb{Z}} \delta(x-y-m).
\] (2.7)

3 Problem 3

In order for the integral to converge, the integrand must become zero at infinity. The integrand may be expressed as

\[
P(t) e^{-t \text{Re}(z)} e^{-t \text{Im}(z)},
\] (3.1)

which becomes zero at infinity if \( \text{Re}(z) > 0 \), since the exponential decay kills off any polynomial contribution. Taking \( P(t) = \sum_{k=0}^{\infty} c_k t^k \), the integral becomes

\[
F(z) = \sum_{k=0}^{\infty} c_k \int_{0}^{\infty} dt t^k e^{-zt}
\] (3.2)

\[
= \sum_{k=0}^{\infty} c_k \int_{0}^{\infty} \frac{dt}{z} \left( \frac{t}{z} \right)^k e^{-t}
\] (3.3)

\[
= \sum_{k=0}^{\infty} c_k z^{-k-1} \Gamma(k+1),
\] (3.4)

which converges for all \( z \neq 0 \).

4 Problem 4

a) Again, rewriting the sum, we have
\[ \xi(s) = 2 \int_0^\infty \frac{dt}{t} t^{\frac{s}{2}} \sum_{n=1}^{\infty} e^{-\pi n^2}. \quad (4.1) \]

Since the sum is absolutely convergent, we may switch the order of the sum and the integration. Then, redefining the integration and using the definition of the gamma function, we obtain

\[ \xi(s) = 2 \sum_{n=1}^{\infty} \int_0^\infty \frac{dt}{\pi n^2} \left( \frac{t}{\pi n^2} \right)^{\frac{s}{2}-1} e^{-t}. \quad (4.2) \]

\[ = 2 \sum_{n=1}^{\infty} \frac{\Gamma(s/2)}{(\pi n^2)^{s/2}} \quad (4.3) \]

\[ = 2 \pi^{-s/2} \Gamma \left( \frac{s}{2} \right) \zeta(s). \quad (4.4) \]

b) According to the suggestion in the problem, we begin by partitioning the integral.

\[ \xi(s) = \int_1^\infty \frac{dt}{t} t^{\frac{s}{2}} \left( \sum_{n \in \mathbb{Z}} e^{-\pi n^2} - 1 \right) + \int_1^\infty \frac{dt}{t} t^{\frac{s}{2}} \left( \sum_{n \in \mathbb{Z}} e^{-\pi n^2} - 1 \right) \quad (4.6) \]

There is no problem with the second integral, but the first integrand diverges as \( t \to 0 \). Using the equivalence provided in the problem statement (0.2) and changing variables \( t \to \frac{1}{t} \), we obtain

\[ \xi(s) = \int_1^\infty \frac{dt}{t} t^{\frac{s}{2}} \left( \frac{1}{\sqrt{t}} \sum_{n \in \mathbb{Z}} e^{-\pi n^2/t} - 1 \right) + \int_1^\infty \frac{dt}{t} t^{\frac{s}{2}} \left( \sum_{n \in \mathbb{Z}} e^{-\pi n^2} - 1 \right) \quad (4.7) \]

\[ = \int_1^\infty \frac{dt}{t} t^{-s/2} \left( \sqrt{t} \sum_{n \in \mathbb{Z}} e^{-\pi n^2} - 1 \right) + \int_1^\infty \frac{dt}{t} t^{\frac{s}{2}} \left( \sum_{n \in \mathbb{Z}} e^{-\pi n^2} - 1 \right) \quad (4.8) \]

\[ = \int_1^\infty \frac{dt}{t} \left( t^{s/2} + t^{(1-s)/2} \right) \left( \sum_{n \in \mathbb{Z}} e^{-\pi n^2} - 1 \right) + \int_1^\infty \frac{dt}{t} \left( t^{(1-s)/2} - t^{-s/2} \right). \quad (4.9) \]

The first integral is now manifestly convergent, while the second is convergent for \( \text{Re}[s] > 1 \). The second integral may be analytically continued by taking \( s \) to be greater than one and defining the result to be the analytic continuation.

\[ \xi(s) = \int_1^\infty \frac{dt}{t} \left( t^{s/2} + t^{(1-s)/2} \right) \left( \sum_{n \in \mathbb{Z}} e^{-\pi n^2} - 1 \right) - \frac{2}{1-s} - \frac{2}{s} \quad (4.10) \]

c) From the result (4.10), it is immediately clear that \( \xi(1-s) = \xi(s) \).

d) Using the results of part (c) and part (a), we obtain

\[ \zeta(1-s) = \frac{1}{2} \frac{\xi(s) \pi^{1-s/2}}{\Gamma \left( \frac{1-s}{2} \right)} \quad (4.11) \]

\[ = \frac{\pi^{1-s/2} \Gamma \left( s/2 \right)}{\Gamma \left( \frac{1-s}{2} \right)} \xi(s). \quad (4.12) \]
e) The residue at $s = 0$ is found immediately from (4.10) to be $-2$.

f) Using the result of part (d), we find

$$\zeta(-1) = \frac{\pi^{-3/2} \Gamma(1)}{\Gamma(-1/2)} \zeta(2) \quad (4.13)$$

$$= -\frac{1}{12}. \quad (4.14)$$

Now, using what we proved in part (a), and recalling that $\Gamma(s) = \frac{1}{s} \Gamma(s + 1)$,

$$\zeta(0) = \lim_{s \to 0} \pi^{s/2} \frac{\Gamma(s/2)}{2 \Gamma(s/2)} \xi(s) \quad (4.15)$$

$$= \frac{1}{4} \lim_{s \to 0} s \xi(s) \quad (4.16)$$

$$= -\frac{1}{2}. \quad (4.17)$$