Exercise 1. Suppose we have a sample of values \(x_1, x_2, \ldots, x_n\). Let \(y_k\) denote the \(k\)th order statistics of this sample. Fix \(p \in (0, 1)\). Write \((n + 1)p = r + \delta\) for a unique integer \(r\) and \(\delta \in [0, 1)\). Show the following:

(i) (# of sample values \(\leq y_r\)) = \(r = np + (p - \delta)\).

(ii) (# of sample values \(> y_r\)) = \(n - r = n(1 - p) - p + \delta\).

Exercise 2. Let \(X_1, X_2, \ldots, X_5\) be i.i.d. \(\text{Exp}(\lambda)\) RVs. Recall that the PDF of \(\text{Exp}(\lambda)\) RVs is given by

\[
f_X(t) = \lambda e^{-\lambda t} 1(t \geq 0).
\]

Let \(Y_1 < Y_2 < \cdots < Y_5\) denote the order statistics of \(X_1, \ldots, X_5\).

(i) Compute the PDFs of \(X_1, \ldots, X_5\).

(ii) Compute the CDFs of \(X_1, \ldots, X_5\).

(iii) Plot the PDFs of \(Y_1 < \cdots < Y_5\).

Exercise 3 (MLE for Geometric RV). Let \(X \sim \text{Geom}(p)\), which is a discrete RV with PMF \(P(X = k) = (1 - p)^{k-1} p, k = 1, 2, 3, \ldots\).

(i) Show that the log likelihood function is given by

\[
l(x_1, \ldots, x_n; p) = n \log p + \left(\sum_{i=1}^{n} x_i\right) - n \log(1 - p).
\]

(ii) Show that the MLE for \(p\) is \(1/\bar{X}\).

Exercise 4 (MLE for Poisson RV). Let \(X \sim \text{Poisson}(\lambda)\), which is a discrete RV with PMF \(P(X = k) = \lambda^k e^{-\lambda} / k!, k = 0, 1, 2, \ldots\).

(i) Show that the log likelihood function is given by

\[
l(x_1, \ldots, x_n; \lambda) = \lambda \sum_{i=1}^{n} x_i - n \lambda + \log(x_1!x_2!\cdots x_n!).
\]

(ii) Show that the MLE for \(\lambda\) is \(\bar{X}\).

Exercise 5 (Sample variance is unbiased). Let \(X_1, \cdots, X_n\) be i.i.d. samples from some distribution with mean \(\mu\) and finite variance. Define the sample variance \(S^2 = (n - 1)^{-1} \sum_{i=1}^{n} (X_i - \bar{X})^2\). We will show that \(S^2\) is an unbiased estimator of the population variance \(\text{Var}(X_1)\).

(i) Show that

\[
\mathbb{E} \left[ \sum_{i=1}^{n} (X_i - \bar{X})(\bar{X} - \mu) \right] = 0.
\]

(ii) Show that

\[
\mathbb{E}[(\bar{X} - \mu)^2] = \mathbb{E} \left[ n^{-2} \left( \sum_{i=1}^{n} (X_i - \mu) \right)^2 \right] = n^{-1} \text{Var}(X_1).
\]

(iii) Show that

\[
\sum_{i=1}^{n} (X_i - \mu)^2 = \sum_{i=1}^{n} (X_i - \bar{X} + (\bar{X} - \mu))^2
\]

\[
= \sum_{i=1}^{n} (X_i - \bar{X})^2 + 2 \sum_{i=1}^{n} (X_i - \bar{X})(\bar{X} - \mu) + n(\bar{X} - \mu)^2.
\]
Taking expectation and using (i), show that

\[ E \left[ \sum_{i=1}^{n} (X_i - \mu)^2 \right] = E \left[ \sum_{i=1}^{n} (X_i - \bar{X})^2 \right] + nE\left[ (\bar{X} - \mu)^2 \right]. \]  
\(8\)

(iv) From (ii) and (iii), deduce that

\[ E \left[ \sum_{i=1}^{n} (X_i - \bar{X})^2 \right] = (n - 1)\text{Var}(X_1). \]  
\(9\)

(v) From (iv), show that

\[ E \left[ S^2 \right] = \text{Var}(X_1). \]  
\(10\)