1. Compute the following limits. If a limit does not exist, be as specific as possible. (E.g. for an infinite limit, find the one-sided limits.) As always, you must justify each answer.

(a) (4 points) \[ \lim_{x \to -2} \frac{-12}{x^2 - 4} - \frac{3}{x + 2} \]

\[ \text{Plugging in gives } \frac{-12}{0} - \frac{3}{0} \]

\[ \text{looks like } \infty - \infty \]

\[ = \lim_{x \to -2} \frac{-12}{(x+2)(x-2)} - \frac{3(x-2)}{(x+2)(x-2)} \]

\[ = \lim_{x \to -2} \frac{-12 - 3(x-2)}{(x+2)(x-2)} \]

\[ = \lim_{x \to -2} \frac{-12 - 3x + 6}{(x+2)(x-2)} \]

\[ = \lim_{x \to -2} \frac{-3x - 6}{(x+2)(x-2)} \]

\[ = \lim_{x \to -2} \frac{-3(x+2)}{(x+2)(x-2)} \]

\[ = \lim_{x \to -2} \frac{-3}{x - 2} = \frac{-3}{-4} = \frac{3}{4} \]

(b) (4 points) \[ \lim_{t \to \frac{\pi}{2}} \frac{t}{\sin(t) - 1} \]

\[ \text{Plugging in gives } \frac{\frac{\pi}{2}}{\sin\left(\frac{\pi}{2}\right) - 1} \]

\[ = \frac{\frac{\pi}{2}}{1-1} = \frac{\text{nonzero}}{0} \]

Infinite limits... check limit from left and from right.

When \( t = \frac{\pi}{2} \), the numerator is \( \approx \frac{\pi}{2} \), so the numerator is positive.
Question 1 continued...

Since \( \sin(t) \leq 1 \) always, \( \sin(t) - 1 \leq 0 \), so the denominator is negative when \( t \approx \frac{\pi}{2} \).

So, from left and from right, we have \( \frac{t}{\sin(t) - 1} = \text{negative} \).

So \( \lim_{t \to \frac{\pi}{2}^+} \frac{t}{\sin(t) - 1} = -\infty \)

and \( \lim_{t \to \frac{\pi}{2}^-} \frac{t}{\sin(t) - 1} = -\infty \).

or just \( \lim_{t \to \frac{\pi}{2}} \frac{t}{\sin(t) - 1} = -\infty \).

(c) (4 points) \( \lim_{x \to 0} \frac{\sin(6x)}{2x} \)

Plugging in gives \( \frac{0}{0} \)

Let \( u = 6x \), so \( x = \frac{u}{6} \), and \( 2x = 2 \frac{u}{6} = \frac{1}{3} u \).

Also as \( x \to 0 \), \( u \to 0 \).

\[
= \lim_{u \to 0} \frac{\sin(u)}{\frac{1}{3} u} \cdot \frac{3}{3}
\]

\[
= 3 \lim_{u \to 0} \frac{\sin(u)}{u} = 3 \cdot 1 = \boxed{3}
\]

Another way: \( \lim_{x \to 0} \frac{\sin(6x)}{2x} = \lim_{x \to 0} \frac{\sin(6x)}{\frac{6}{3}(6x)} \)

\[
= 3 \lim_{x \to 0} \frac{\sin(6x)}{6x}
\]

\[
= 3 \lim_{u \to 0} \frac{\sin(u)}{u} = 3 \cdot 1 = \boxed{3}
\]
2. (a) (3 points) Complete the following definition:

Suppose \( f(x) \) is defined on an open interval containing \( x = a \). Then \( f \) is continuous at \( x = a \) if

\[
\lim_{x \to a} f(x) = f(a)
\]

(b) (9 points) Define a function \( f: \mathbb{R} \to \mathbb{R} \) as follows, where \( c \) is a constant:

\[
f(x) = \begin{cases} 
\frac{\sqrt{x^2+5x-6}}{x^2-9x+20} & \text{if } x < 4 \\
\frac{x^2-cx-9}{2x+4} & \text{if } x \geq 4 
\end{cases}
\]

What value of \( c \) is needed to make \( f \) a continuous function?

\[
\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} \frac{x^2-cx-9}{2x+4} = \frac{16-4c-9}{12} = \frac{7-4c}{12}
\]

Plugging in gives \( \frac{0}{0} \)

\[
\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} \frac{\sqrt{x^2+5x-6}}{x^2-9x+20} \cdot \frac{(x+9)(x-4)}{(x-5)(x-4)(\sqrt{x^2+5x+6})} 
\]

\[
= \lim_{x \to 4^-} \frac{x^2+5x-36}{(x-5)(x-4)(\sqrt{x^2+5x+6})} 
\]

\[
= \lim_{x \to 4^-} \frac{x+9}{(x-5)(x-4)} = \frac{13}{(-1)(6+6)} = \frac{-13}{12}
\]

To be continuous, we need \( \frac{7-4c}{12} = \frac{-13}{12} \)

So \( 7-4c = -13 \), so \( 4c = 20 \), so \( c = 5 \)
(Solution to #2, continued...)

What we have found here is that when \( c = 5 \),
\[
\lim_{{x \to 4^+}} f(x) = \lim_{{x \to 4^-}} f(x) = -\frac{13}{12},
\]
and therefore \( \lim_{{x \to 4}} f(x) \) exists.

But furthermore, from the definition of \( f \),
\[
f(4) = \frac{x^2 - 5x - 9}{2x + 4} \bigg|_{{x = 4}} = -\frac{13}{12}.
\]

Therefore, when \( c = 5 \),
\[
\lim_{{x \to 4}} f(x) = f(4),
\]
so \( f \) is continuous at \( x = 4 \).

Note also that the function \( \frac{\sqrt{x^2 + 5x} - 6}{x^2 - 9x + 20} = \frac{\sqrt{x^2 + 5x} - 6}{(x-5)(x-4)} \)
has another discontinuity (vertical asymptote in fact) at \( x = 5 \),
however this does not affect \( f \), since \( f(x) \) is equal to this function only for \( x < 4 \).

Likewise, the function \( \frac{x^2 - 5x - 9}{2x + 4} \) has a discontinuity (vertical asymptote) at \( x = -2 \), but again this does not affect \( f \),
because \( f(x) \) is equal to this function only for \( x \geq 4 \).

So in fact, when \( c = 5 \), \( f \) is continuous everywhere.
3. The two parts of this problem are not related to each other.

(a) (6 points) Let \( g(x) = 3x^{\frac{3}{2}} + \frac{16}{\sqrt{x}} - 5 \).

Find an equation for the tangent line to the graph of \( g(x) \) at \( x = 4 \).

\[
\frac{dg}{dx} = \frac{d}{dx} \left( 3x^{\frac{3}{2}} + \frac{16}{\sqrt{x}} - 5 \right)
= 3 \cdot \frac{3}{2} x^{-\frac{1}{2}} + 16 \cdot \frac{1}{x^{\frac{3}{2}}} - \frac{d}{dx}(5)
= 3 \cdot \frac{3}{2} x^{-\frac{1}{2}} + 16 \cdot (-\frac{1}{2} x^{-\frac{3}{2}}) - 0
= \frac{9}{2} \sqrt{x} - 8 \cdot \frac{1}{\sqrt{x}}
\]

Slope of tangent line: \( \left. \frac{dg}{dx} \right|_{x=4} = \frac{9}{2} \sqrt{4} - 8 \cdot \frac{1}{\sqrt{4}} = \frac{9 \sqrt{4}}{2} - 8 = 8 \)

Tangent line: \( y = 8 \cdot (x - 4) + 27 \) \( \Rightarrow \) point-slope form

(b) (6 points) Let \( h(x) = x^3 - 3x^2 - 6x + 4 \).

At what \( x \)-value(s) does the graph of \( h(x) \) have a tangent line parallel to \( y = 3x - 5 \)?

Need tangent line to have slope 3.

Slope of tangent line: \( \frac{dh}{dx} = \frac{d}{dx} \left( x^3 - 3x^2 - 6x + 4 \right) = 3x^2 - 3 \cdot 2x - 6 \cdot 1 + 0 = 3x^2 - 6x - 6 \).

So we want \( \frac{dh}{dx} = 3 \)

\[ \begin{align*}
3x^2 - 6x - 6 &= 3 \\
3x^2 - 6x - 9 &= 0 \\
x^2 - 2x - 3 &= 0
\end{align*} \]

\( (x-3)(x+1)=0 \)

\( x = 3 \) and \( x = -1 \)
4. (9 points) Let \( f(x) = \sqrt{x^3} \). Use the limit definition of the derivative* to compute \( f'(1) \).

\[
f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{\sqrt{(1+h)^3} - \sqrt{1^3}}{h} = \lim_{h \to 0} \frac{(\sqrt{(1+h)^3} - 1)(\sqrt{(1+h)^3} + 1)}{h(\sqrt{(1+h)^3} + 1)} \quad \text{Plugging in gives } 0 \]

\[
= \lim_{h \to 0} \frac{(1+h)^3 - 1}{h(\sqrt{(1+h)^3} + 1)}
\]

\[
= \lim_{h \to 0} \frac{1 + 3h + 3h^2 + h^3 - 1}{h(\sqrt{(1+h)^3} + 1)}
\]

\[
= \lim_{h \to 0} \frac{h(3 + 3h + h^2)}{h(\sqrt{(1+h)^3} + 1)}
\]

\[
= \lim_{h \to 0} \frac{3 + 3h + h^2}{\sqrt{(1+h)^3} + 1} = \frac{3}{\sqrt{1^3} + 1} = \frac{3}{2}
\]

*You may use either of the two formulas we learned for the limit definition of the derivative. Be sure to show all of your steps. Do not just use differentiation rules! Although you can use those to double-check your answer, of course.