0.0.0.1 Problem 2.2.4

Exercise 2.2.4 Notice that our calculation results in a negative number. Why does this make sense?

In this case, as time passes, the height decreases (since the object is falling). Since $H(t_2) < H(t_1)$ (for $t_2 > t_1$), the object is heading in a “negative” direction, which can only be reflected via a negative rate of change. Mathematically, we know that:

$$t_2 > t_1 \iff t_2 - t_1 > 0 \iff \Delta t > 0$$
$$H(t_2) < H(t_1) \iff H(t_2) - H(t_1) > 0 \iff \Delta H < 0$$

From there, we have:

$$\text{average rate of change} = \frac{\Delta H}{\Delta t} = \begin{cases} \text{Negative Number as } \Delta H < 0 \\ \text{Positive Number as } \Delta t > 0 \end{cases} = \text{Negative Number < 0}$$
Problem 2.2.5

Exercise 2.2.5  Approximate $H'(1.5)$ using the time interval $\Delta t = 0.0001$.

We have the $H(t)$ equation:

$$H(t) = H(0) - 16t^2 = 100 - 16t^2$$

Using $\Delta t = 0.0001$, our approximation for $H'(1.5)$ is:

$$H'(1.5) \approx \frac{H(t + \Delta t) - H(t)}{\Delta t} = \frac{H(1.5001) - H(1.5)}{0.0001}$$

$$= \frac{(100 - 1.5001^2) - (100 - 1.5^2)}{0.0001} = \frac{-48.0016}{\text{s}}$$
Problem 2.2.6

Exercise 2.2.6 Why do we not allow \( \Delta t \) to reach 0?

We know that for a function \( f(t) \), the average rate of change of \( f \) at \( t = t_0 \) can be approximated using a small \( \Delta t \), where:

\[
\text{average rate of change of } f \text{ at } (t = t_0) \approx \frac{f(t_0 + \Delta t) - f(t_0)}{\Delta t}
\]

Now, if \( \Delta t = 0 \), we can clearly see that the above equation will become:

\[
\text{average rate of change of } f \text{ at } (t = t_0) \approx \frac{f(t_0 + 0) - f(t_0)}{0} = \frac{f(t_0) - f(t_0)}{0} = \frac{0}{0}
\]

No one knows what the fraction \( \frac{0}{0} \) will be equal to. Is it 0 or is it undefined?

Thus, to successfully obtain a definite (calculate-able) value for the average rate of change, we cannot allow \( \Delta t \) to reach 0.
Exercise 2.2.7 Use successive approximations to find the object’s speed at \( t = 1 \) second.

We have:

\[
H(t) = H(0) - 16t^2 = 100 - 16t^2
\]

To approximate \( H' \) at \( t = 1 \), we will use the following \( \Delta t \): 0.1, 0.01, 0.001, 0.0001, and 0.00001. We know that the equation to approximate \( H'(1) \) is:

\[
H'(1) \approx \frac{H(t + \Delta t) - H(t)}{\Delta t} = \frac{H(1 + \Delta t) - H(1)}{\Delta t} = \frac{[100 - 16 \cdot (1 + \Delta t)^2] - [100 - 16 \cdot (1)^2]}{\Delta t}
\]

Using the above equation, we obtain the following results:

\[
\begin{align*}
\Delta t = 0.1 \rightarrow H'(1) &\approx \frac{[100 - 16 \cdot (1 + 0.1)^2] - [100 - 16 \cdot (1)^2]}{0.1} = -33.6 \\
\Delta t = 0.01 \rightarrow H'(1) &\approx \frac{[100 - 16 \cdot (1 + 0.01)^2] - [100 - 16 \cdot (1)^2]}{0.01} = -32.16 \\
\Delta t = 0.001 \rightarrow H'(1) &\approx \frac{[100 - 16 \cdot (1 + 0.001)^2] - [100 - 16 \cdot (1)^2]}{0.001} = -32.016 \\
\Delta t = 0.0001 \rightarrow H'(1) &\approx \frac{[100 - 16 \cdot (1 + 0.0001)^2] - [100 - 16 \cdot (1)^2]}{0.0001} = -32.0016 \\
\Delta t = 0.00001 \rightarrow H'(1) &\approx \frac{[100 - 16 \cdot (1 + 0.00001)^2] - [100 - 16 \cdot (1)^2]}{0.00001} = -32.00016
\end{align*}
\]

As \( \Delta t \) approaches 0, we can see that \( H'(1) \) approaches the value of \(-32\). Thus, we approximate the object’s speed at \( t = 1 \) second to be:

\[
H'(1) = -32 \text{ ft/s}
\]
0.0.0.1 Problem 2.2.8

Exercise 2.2.8 Carry out a similar calculation for \( t = 2 \).

We have the equation:

\[
H(t) = H(0) - 16t^2 = 100 - 16t^2 \tag{1}
\]

Using Equation (1), we obtain the following expressions:

\[
H(2.0) = 100 - 16 \cdot (2.0)^2 = 36 \tag{2}
\]

\[
H(2.0 + \Delta t) = 100 - 16 \cdot (2.0 + \Delta t)^2
= 100 - 16 \cdot [(2.0)^2 + 2 \cdot (2.0) \cdot (\Delta t) + (\Delta t)^2]
= 100 - 16 \cdot [4 + 4 \cdot (\Delta t) + (\Delta t)^2] \tag{3}
= 100 - 64 - 64 \cdot (\Delta t) - 16 \cdot (\Delta t)^2
= 36 - 64 \cdot (\Delta t) - 16 \cdot (\Delta t)^2
\]

Then, we know the equation for the average rate of change to be:

\[
\text{average rate of change at } (t = 2.0) = \frac{H(2.0 + \Delta t) - H(2.0)}{\Delta t} \tag{4}
\]

Substituting the Equations (2) and (3) into Equation (4), we have:

\[
\text{average rate of change at } (t = 2.0) = \frac{\left( 36 - 64 \cdot (\Delta t) - 16 \cdot (\Delta t)^2 \right) - 36}{\Delta t}
= \frac{-64 \cdot (\Delta t) - 16 \cdot (\Delta t)^2}{\Delta t} \tag{5}
= \frac{(\Delta t) \cdot (-64 - 16 \cdot \Delta t)}{\Delta t}
= -64 - 16 \cdot \Delta t
\]

We have now found a general expression for the average rate of change at \( t = 2.0 \) as a function of \( \Delta t \). We can also clearly see that, as \( \Delta t \to 0 \), \((-64 - 16 \cdot \Delta t) \to -64 \). Thus:

\[
H'(2.0) = \lim_{\Delta t \to 0} \left[ \frac{H(2.0 + \Delta t) - H(2.0)}{\Delta t} \right] = \lim_{\Delta t \to 0} (-64 - 16 \cdot \Delta t) \iff H'(2.0) = -64
\]
0.0.0.1 Problem 2.2.9

**Exercise 2.2.9** Use this result to find the object’s velocity at $t = 2$.

From the paragraph above, we know that we calculate the velocity of the object at any time $t$ using the formula:

$$H'(t) = -32t$$

Thus, the object’s velocity at $t = 2$ is:

$$H'(2) = -32 \cdot 2 \iff H'(2) = -64 \frac{\text{ft}}{\text{s}}$$
0.0.0.1 Problem 2.2. Further Exercise 1

**Exercise 2.2.FE 1**  The rate of change of the position of a car at some time \( t_0 \) is given by \( \frac{dX}{dt} = 55 \). What does this mean in plain English?

This means that at time \( t_0 \), the car is moving at a speed of 55, in that very instant.
Exercise 2.2.FE 2 You are studying the athletic performance of runners. You have two motion-triggered cameras that produce time-stamped photographs.

a) The runner reaches the first camera, at the 500 m mark, at 9:03:05 a.m. and the second camera, at the 600 m mark, at 9:03:25 a.m. What is her average speed over that time interval?

b) When is she running at that speed?

c) How could you change your measurement setup (without getting new equipment) to better approximate the runner’s instantaneous speed at 500 m?

Part a: The runner reaches the first camera, at the 500 m mark, at 9:03:05 a.m. and the second camera, at the 600 m mark, at 9:03:25 a.m. What is her average speed over that time interval?

As we observed, the runner travels for about 100 m (from the 500 m mark to 600 m mark) within 20 seconds (between 9:03:05 am and 9:03:25 am).

Thus, her average speed over that time interval is:

\[
\text{average speed} = \frac{\text{Distance Travel}}{\text{Time}} = \frac{600 - 500}{9:03:25 \text{ am} - 9:03:05 \text{ am}} = \frac{100}{20} = \frac{5 \text{ m}}{\text{s}}
\]

Part b: When is she running at that speed?

The answer is: we don’t know for sure... She could have run at 5 m/s at any given time between 9:03:05 a.m. and 9:03:25 a.m., but we do NOT know exactly when.

Part c: How could you change your measurement setup (without getting new equipment) to better approximate the runner’s instantaneous speed at 500 m?

To do this, we need to “shrink \( \Delta t \)”. And one way to shrink \( \Delta t \) is to measure her speed over a shorter distance (since she will need less time to cover a shorter distance). As a result, to better approximate the runner’s instantaneous speed at 500 m,

We will move the motion-triggered camera, originally at the 600 m mark, closer and closer to the 500 m mark.
You are an ecologist studying bottom-dwelling stream invertebrates. You need to measure the speed at which the water is flowing at a particular point you have chosen to study. You have a stopwatch, a long measuring tape, a supply of Ping-Pong balls (which float and are easy to see), and brightly colored flags that can be used to mark points along the shore or in the water. How would you use this equipment to estimate the instantaneous speed of the water? (You may want to include a diagram with your response.)

Let’s draw up a diagram for now.

As we can see, one way that we can measure the speed of the water is:

1. Set up the Start and Finish flags as shown in the picture.
2. Float the Ping Pong Balls on the river. The speed of the ping pong balls will be the same as the speed of the water.
3. When the ping pong ball hits the Start line, record the location $X_0$ and time $t_0$.
4. Then, when the ping pong ball hits the Finish line, record the location $X_f$ and time $t_f$.
5. Calculate the speed of the ping pong ball, hence the speed of the water:

$$\text{Average Speed} = \frac{\Delta X}{\Delta t} = \frac{X_f - X_0}{t_f - t_0}$$

Now, to measure the instantaneous speed of the water, we need to “shorten the $\Delta t$”. One way that we can do that is to measure over a shorter distance (hence the water needs less time to cover a
shorter distance). In other words, in our case, we just have to move the Finish flag closer to the Start flag (which marks our Point of Interest) and repeat the exercise. As the flags come closer together, we can see the “average speed” approaching a certain value, i.e. the actual instantaneous speed of the water at the point of interest.

Visually, we have:

![Diagram showing river flow with flags](image)

*However, this is not the only acceptable approach. Other equally acceptable approaches include:
1. Making the Point of Interest at the Finish Line*
2. Put the *Start* and *Finish* Flags on either side of the *Point of Interest*:
0.0.0.1 Problem 2.3.1

**Exercise 2.3.1** Calculate the slope of the secant line to the graph of $Y = \frac{X}{1+X}$ from $X = 1$ to $X = 3$.

We have:

$$\text{slope of the secant line} = \frac{\Delta Y}{\Delta X} = \frac{Y(3) - Y(1)}{3 - 1} = \frac{3}{1+3} - \frac{1}{1+1} = \frac{1}{8} = 0.125$$
0.0.0.1 Problem 2.3.2

**Exercise 2.3.2** Find the slope of the secant line crossing the graph of $f(t) = 200 - 16t^2$ at the following values of $t$. What value is the slope approaching?

a) $t = 2, t = 2.5$  

b) $t = 2, t = 2.1$  

c) $t = 2, t = 2.05$

**Part a: $t = 2, t = 2.5$**

We have:

$$\text{slope} = \frac{\Delta f}{\Delta t} = \frac{f(2.5) - f(2)}{2.5 - 2} = \frac{\left(200 - 16 \cdot (2.5)^2\right) - \left(200 - 16 \cdot 2^2\right)}{2.5 - 2} = -72$$

**Part b: $t = 2, t = 2.1$**

We have:

$$\text{slope} = \frac{\Delta f}{\Delta t} = \frac{f(2.1) - f(2)}{2.1 - 2} = \frac{\left(200 - 16 \cdot (2.1)^2\right) - \left(200 - 16 \cdot 2^2\right)}{2.1 - 2} = -65.6$$

**Part c: $t = 2, t = 2.05$**

We have:

$$\text{slope} = \frac{\Delta f}{\Delta t} = \frac{f(2.05) - f(2)}{2.05 - 2} = \frac{\left(200 - 16 \cdot (2.05)^2\right) - \left(200 - 16 \cdot 2^2\right)}{2.05 - 2} = -64.8$$

**Approaching Value:**

The slope is approaching the value of $-64$. 
Exercise 2.3.3 Find the equation of the tangent line to $f(t) = 200 - 16t^2$ at $t = 2$.

We know that for $f(t)$, the equation for the tangent line to $f(t)$ at $t = t_0$ is:

$$f(t) - f(t_0) = \left. \frac{df}{dt} \right|_{t_0} (t - t_0)$$

We have that at $t = 2$,

$$f(2) = 200 - 16 \cdot 2^2 = 200 - 16 \cdot 4 = 200 - 64 = 136$$

$$\left. \frac{df}{dt} \right|_{t_0} = f'(2) = -64 \text{ (from Exercise 2.3.2)}$$

Thus, the equation of the tangent line to $f(t) = 200 - 16t^2$ at $t = 2$ is:

$$f(t) - 136 = (-64)(t - 2) \iff f(t) - 136 = (-64)(t - 2) \iff f(t) = (-64)t + 264$$
0.0.0.1 Problem 2.3.4

**Exercise 2.3.4** Write equations for the following lines in both slope-intercept and point-slope form.

a) The line that has a slope of 2 and a \( Y \)-intercept of \(-54\).

b) The line that has a slope of \(-3\) and passes through the point \((2, 6)\).

c) The line that passes through the points \((1, 7)\) and \((3, 5)\).

For a line with slope \( m \) that passes through point \((X_0, Y_0)\), we have the equation representing the line expressed in the following forms:

<table>
<thead>
<tr>
<th>Slope-Intercept Form</th>
<th>Point-Slope Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Y = m \cdot X + b )</td>
<td>( Y - Y_0 = m \cdot (X - X_0) )</td>
</tr>
</tbody>
</table>

**Part a: The line that has a slope of 2 and a \( Y \)-intercept of \(-54\).**

**Point-Slope Form:**

One way to think about this question is that this line has a slope of 2 and cuts the \( Y \)-axis at point \((0, -54)\). Thus, we have:

- \( m = 2 \)
- \( X_0 = 0 \)
- \( Y_0 = -54 \)

As a result, the point-slope form is:

\[
Y - Y_0 = m \cdot (X - X_0) \iff Y - (-54) = 2 \cdot (X - 0) \iff Y + 54 = 2 \cdot (X - 0)
\]

**Slope-Intercept Form:**

From the point-slope form, we will derive the slope-intercept form:

\[
Y + 54 = 2 \cdot (X - 0) \iff Y = 2 \cdot X - 54
\]

**Part b: The line that has a slope of \(-3\) and passes through the point \((2, 6)\).**

**Point-Slope Form:**

We have:
\[ m = -3 \]
\[ X_0 = 2 \]
\[ Y_0 = 6 \]

As a result, the point-slope form is:
\[ Y - Y_0 = m \cdot (X - X_0) \iff Y - (6) = (-3) \cdot (X - 2) \iff Y - 6 = (-3) \cdot (X - 2) \]

**Slope-Intercept Form:**
From the point-slope form, we will derive the slope-intercept form:
\[ Y - 6 = (-3) \cdot (X - 2) \iff Y = (-3) \cdot X + 12 \]

**Part c: The line that passes through the points (1, 7) and (3, 5).**

**Point-Slope Form:**
For our problem, we will define:

- The first point \((X_1, Y_1)\) to be \((1, 7)\).
- The second point \((X_2, Y_2)\) to be \((3, 5)\).

Let’s first calculate the slope. We have:
\[ m = \frac{Y_2 - Y_1}{X_2 - X_1} = \frac{5 - 7}{3 - 1} = \frac{-2}{2} = -1 \]

Now, depending on what you pick as your point of interest, we can have one of the following two answers.

**Choosing Point \((1, 7)\):**
We have:

- \(m = -1\)
- \(X_0 = 1\)
- \(Y_0 = 7\)

As a result, the point-slope form is:
\[ Y - Y_0 = m \cdot (X - X_0) \iff Y - (7) = (-1) \cdot (X - 1) \iff Y - 7 = (-1) \cdot (X - 1) \]

**Choosing Point \((3, 5)\):**
We have:
- $m = -1$
- $X_0 = 3$
- $Y_0 = 5$

As a result, the point-slope form is:

$$Y - Y_0 = m \cdot (X - X_0) \iff Y - (5) = (-1) \cdot (X - 3) \iff Y - 5 = (-1) \cdot (X - 3)$$

**Slope-Intercept Form:**

Regardless, of what was actual your point-slope form, your slope-intercept form can only have one answer, which we can show below.

$$Y - 7 = (-1) \cdot (X - 1) \iff Y = (-1) \cdot X + 8$$

$$Y - 5 = (-1) \cdot (X - 3) \iff Y = (-1) \cdot X + 8$$
### Problem 2.3. Further Exercise 1

**Exercise 2.3.FE 1** If for some function \( f \), \( f(2) = 5 \) and \( f'(2) = -3 \), what is the tangent line to \( f \) at \( X = 2 \)?

You can express the tangent line in *either format*: slope-intercept or point-slope.

We have the following data:

- \( f'(2) = -3 \iff m = -3 \)
- \( f(2) = 5 \iff X_0 = 2 \), and \( Y_0 = 5 \)

As a result, the equation of the tangent line is:

\[
Y - Y_0 = m \cdot (X - X_0)
\]

\[
\iff Y - 5 = (-3) \cdot (X - 2) \text{ (point-slope form)}
\]

\[
\iff Y = (-3) \cdot X + 11 \text{ (slope-intercept form)}
\]
0.0.0.1 Problem 2.3. Further Exercise 2

**Exercise 2.3.FE 2** If some function $f$ has the tangent line $y - 2 = 4(t - 16)$ at the point implied by the equation, what are $f(16)$ and $f'(16)$?

We know that the equation for the tangent line that goes through $(x_0, y_0)$ and has a slope of $m = \frac{dy}{dx} \bigg|_{x_0}$ is:

$$y - y_0 = m \cdot (x - x_0) = \frac{dy}{dx} \bigg|_{x_0} \cdot (x - x_0)$$

Now, let’s look at our equation for the tangent line to $f$ and label the appropriate parts:

$$y - 2 = 4 \left( t - \frac{16}{x_0} \right)$$

Thus, we know that:

$$f(16) = f(x_0) = y_0 \iff f(16) = 2$$

$$f'(16) = m = \frac{dy}{dx} \bigg|_{x_0} = \frac{dy}{dx} \bigg|_{x=16} \iff f'(16) = 4$$
**Exercise 2.4.3**  What is the complete equation for the tangent line to $Y = f(X)$ at the point $(X_0, f(X_0))$?

We know that equation of a line that passes through point $(X_0, Y_0)$ and has a slope of $m = \frac{dY}{dX} \bigg|_{X=X_0}$ is (in point-slope form):

$$Y - Y_0 = \frac{dY}{dX} \bigg|_{X=X_0} \cdot (X - X_0)$$

We have the following data:

- $X_0 = X_0$
- $Y_0 = f(X_0)$
- $m = \frac{df}{dX} \bigg|_{X=X_0} = f'(X_0)$

Thus, the complete equation for the tangent line to $Y = f(X)$ at the point $(X_0, f(X_0))$ is:

$$Y - f(X_0) = f'(X_0) \cdot (X - X_0)$$
Exercise 2.4.5  In the example of the falling object, we calculated its velocity $H'(1.5)$, the rate of change of height with respect to time, at 1.5 seconds after it was released. We got the answer $-48 \text{ ft/s}$. Now estimate how far the ball will drop in the next 0.01 seconds. In other words, let $\Delta t = 0.01$ seconds, and calculate an approximate value for $\Delta H$.

We know that for small $\Delta t$,

\[
\Delta H \approx \left. \frac{dH}{dt} \right|_{t_0} \cdot \Delta t = (-48) \cdot (0.01) = -0.48 \text{ ft}
\]

In other words, the ball drops approximately 0.48 ft in height.
0.0.0.1 Problem 2.4.6

Exercise 2.4.6 The equation for the height of the falling ball is

\[ H(t) = H(0) - 16t^2 \]

Use this equation to calculate the actual change in \( H \) from \( t = 1.5 \) s to \( t = 1.51 \) s. How close is this actual \( \Delta H \) to the \( \Delta H \) you calculated in Exercise 2.4.5?

Using the equation \( H(t) = H(0) - 16t^2 \), we can calculate the actual \( \Delta H \) to be:

\[
\Delta H = H(1.51) - H(1.50) = \frac{100 - 16 \cdot (1.51)^2}{H(1.51)} - \frac{100 - 16 \cdot (1.50)^2}{H(1.50)} \quad \Longrightarrow \quad \Delta H = -0.4816 \text{ ft}
\]

Considering that we approximated \( \Delta H \approx -0.48 \) ft in Exercise 2.4.5, the actual result is very close to our approximation. More precisely, the difference between the answers is only:

\[
|\Delta H_{\text{actual}} - \Delta H_{\text{approx.}}| = |(-0.4816) - (-0.48)| = |0.0016| = 0.0016
\]