Problem 2.4. Further Exercise 1

Exercise 2.4.FE 1 You are studying a new blood pressure drug. At a dose of 5 mg, the slope of the dose-response curve is $-2 \frac{\text{mmHg}}{\text{mg}}$. Approximately how much would a patient’s blood pressure change if the drug dose was increased to 5.1 mg?

Let’s first take a look at the slope to make sure that we get the right order of division. We have:

$$-2 \frac{\text{mmHg}}{\text{mg}} \iff \text{we have a } \frac{\text{d(Pressure)}}{\text{d(Dose)}}$$

If we define $P \equiv \text{Pressure}$, and $D \equiv \text{Drug Dose}$, then we have: $\frac{dP}{dD} = -2$

Now, we have the following expression that we want to solve for $\Delta P$:

$$\frac{\Delta P}{\Delta D} \approx \frac{dP}{dD} \bigg| _{D=D_0} \iff \Delta P \approx \frac{dP}{dD} \bigg| _{D=D_0} \cdot \Delta D = (-2) \cdot (5.1 - 5.0) \iff \Delta P = -0.2 \text{ mmHg}$$

In other words, if the drug dose was increased to 5.1 mg,

**a patient’s blood pressure will drop by 0.2 mmHg.**
Exercise 2.4.FE 2  Suppose \( g(N) \) measures the size of tomatoes produced by a tomato plant as a function of the amount \( N \) of nitrogen that is available to the plant.

a) Explain in plain English (without using the word “derivative”) what the quantity \( \frac{dg}{dN} \) means.

b) If at some instant \( \frac{dg}{dN} \) was equal to 5, and \( N \) was then increased by 0.04, what would you expect to happen to \( g \)? Be as specific as possible.

Part a: Explain in plain English (without using the word “derivative”) what the quantity \( \frac{dg}{dN} \) means.

The quantity \( \frac{dg}{dN} \) means: how fast we can change the size of tomatoes, based on how much we change the amount of nitrogen that is available to the plant.

Part b: If at some instant \( \frac{dg}{dN} \) was equal to 5, and \( N \) was then increased by 0.04, what would you expect to happen to \( g \)? Be as specific as possible.

We know that:

\[
\frac{\Delta g}{\Delta N} \approx \frac{dg}{dN} \bigg|_{N=N_0} \iff \Delta g \approx \frac{dg}{dN} \bigg|_{N=N_0} \cdot \Delta N = 5 \cdot 0.04 \iff \Delta g = 0.20
\]

In other words, \( \Delta g \) will \textbf{INCREASE} by 0.20.
0.0.0.1 Problem 2.4. Further Exercise 4

**Exercise 2.4.FE 4** Sketch graphs of functions that match the following descriptions:

a) The function is discontinuous at $X = 2$ but continuous everywhere else.

b) The function is continuous at $X = 5$ but has no tangent line there.

c) The function is not differentiable at $X = 1$ but has a tangent line there.

*(Please note that there is more than one possible set of solutions to this problem)*

**Part a:** The function is discontinuous at $X = 2$ but continuous everywhere else.

The main idea is that there is a **gap** between the two segments that represent the function, and this gap occurs at $X = 2$.

**Part b:** The function is continuous at $X = 5$ but has no tangent line there.

The main idea is that there is either a **corner** or **cusp** at $X = 5$. We cannot draw a tangent line at a corner or a cusp.
**Part c:** The function is not differentiable at $X = 1$ but has a tangent line there.

The main idea is that there is either a **vertical tangent line at $X = 1$**.
Exercise 2.5.1  Match each function \( f \) in the top row to its derivative \( f' \) in the bottom row. We have done the first one for you. Make sure you understand this, and then match the others.

Let’s walk through one particular graph together to establish a solid methodology that we will use for the rest of the graphs. In this case, we will do \( f_3 \) (graph #3 for \( f \)). We will recreate a similar version to #3 with defined values, that we can talk use to talk about cleanly.

For each of the green point on the graph, we draw a tangent line to the graph of \( f(x) \) at that point. We all know that at point \( x = x_0 \), the slope of the tangent line is determined by:

\[
m = f'(x_0)
\]

Thus, based on the graph, we can see that if:

- \( m > 0 \iff f'(x_0) > 0 \)
• \( m = 0 \iff f'(x_0) = 0 \)

• \( m < 0 \iff f'(x_0) < 0 \).

Based on this information, we can know find the sign of \( f'(x_0) \) based on the sign of \( m \).

Thus, we have our answer:

<table>
<thead>
<tr>
<th>Graph of ( f(x) )</th>
<th>Corresponding Graph of ( f'(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Graph of ( f(x) )" /></td>
<td><img src="image2" alt="Corresponding Graph of ( f'(x) )" /></td>
</tr>
</tbody>
</table>

Using this methodology, we have the following answers to our problem:
### Recognizing the Patterns for $f'(x)$

<table>
<thead>
<tr>
<th>Graph of $f(x)$</th>
<th>Recognizing the Patterns for $f'(x)$</th>
<th>Corresponding Graph of $f'(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Graph 1" /></td>
<td>Regardless of your $x$ value, all of the tangent lines of this graph will have the same slope, and that slope will be positive. Thus, we are looking for a graph of $f'(x)$ that is a flat line (i.e. no change in value of $f'(x)$) that also cuts the vertical axis at a positive value.</td>
<td><img src="image2" alt="Graph C" /></td>
</tr>
<tr>
<td><img src="image3" alt="Graph 2" /></td>
<td>For $x &lt; 0$, the graph is going down, i.e. $f'(x) &lt; 0$. At $x = 0$, it flattens out, i.e. $f'(0) = 0$. Then, it starts to go up for $x &gt; 0$, i.e. $f'(x) &gt; 0$.</td>
<td><img src="image4" alt="Graph A" /></td>
</tr>
<tr>
<td><img src="image5" alt="Graph 3" /></td>
<td>We did this particular example above, but briefly, we will go from left to right. Starting with values of $x &lt; 0$, the graph increases (i.e. $f'(x) &gt; 0$) until $x$ reaches a certain value of $\alpha &lt; 0$. At $x = \alpha$, the graph flattens out (i.e. $f'(\alpha) = 0$), then goes down (i.e. $f'(x) &lt; 0$). Then, as $x$ reaches a value of $\beta &gt; 0$, the graph flattens out again (i.e. $f'(\beta) = 0$), before finally going up (i.e. $f'(x) &gt; 0$).</td>
<td><img src="image6" alt="Graph B" /></td>
</tr>
<tr>
<td><img src="image7" alt="Graph 4" /></td>
<td>The graph is always going down, thus the entire $f'(x)$ graph will lie below the $x$–axis. Now, if we look closely, we will see that: on the left side, the graph of $f(x)$ is going down faster than on the right side. Thus, $f'(x)$ is more negative on the left side, than on the right side.</td>
<td><img src="image8" alt="Graph E" /></td>
</tr>
<tr>
<td><img src="image9" alt="Graph 5" /></td>
<td>The graph is always going up, thus the entire $f'(x)$ graph will lie above the $x$–axis. Now, if we look closely, we will see that: on the left side, the graph of $f(x)$ is going up slower than on the right side. Thus, $f'(x)$ is less positive on the left side, than on the right side.</td>
<td><img src="image10" alt="Graph D" /></td>
</tr>
</tbody>
</table>
Exercise 2.5.2  What does “the derivative of $f(x)$ is $7x + 4.5$” mean? Give two answers.

The statement of “the derivative of $f(x)$ is $7x + 4.5$” can mean a few things:

- At $x$, the rate of change of the function $x$, relative to $x$, can be calculated using the formula: $7x + 4.5$.

- The slope to the tangent line to $f(x)$ at any point $x$ can be calculated using the formula: $7x + 4.5$.

- At every point $x$, $\Delta f = (7x + 4.5) \cdot \Delta x$ is the linear approximation to $f(x)$. 
Problem 2.5.3

Exercise 2.5.3  Find the derivative of the function $f(X) = X^3$ as in the above example. (Recall from algebra that $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.)

First, let’s plug the definition of $f$ into the expression for the average change:

$$
\frac{f(X + \Delta X) - f(X)}{\Delta X} = \frac{(X + \Delta X)^3 - (X)^3}{\Delta X}
$$

$$
= \frac{X^3 + 3 \cdot (X)^2 \cdot (\Delta X) + 3 \cdot (X) \cdot (\Delta X)^2 + (\Delta X)^3 - X^3}{\Delta X}
$$

$$
= \frac{(\Delta X) \cdot [3 \cdot (X)^2 + 3 \cdot (X) \cdot (\Delta X) + (\Delta X)^2]}{\Delta X}
$$

$$
= 3 \cdot (X)^2 + 3 \cdot (X) \cdot (\Delta X) + (\Delta X)^2
$$

Letting $\Delta X$ approaches 0, we obtain the derivation of our function:

$$
f'(X) = \lim_{\Delta X \to 0} \left[ \frac{f(X + \Delta X) - f(X)}{\Delta X} \right] = \lim_{\Delta X \to 0} \left[ 3 \cdot (X)^2 + 3 \cdot (X) \cdot (\Delta X) + (\Delta X)^2 \right]
$$

$$
\iff f'(X) = 3 \cdot X^2
$$
0.0.0.1 Problem 2.5.4

Exercise 2.5.4 Calculate the slope of the tangent line to the graph of \( Y = X^3 \) at \( X = 1 \).

From Problem 2.5.3, we found the derivative for \( f(X) = X^3 \) to be: \( f'(X) = 3 \cdot X^2 \)

Thus, the slope of the tangent line to the graph of \( Y = X^3 \) at \( X = 1 \) is:

\[
f'(1) = 3 \cdot (1)^2 \quad \iff \quad f'(1) = 3
\]
0.0.0.1 Problem 2.5.5

Exercise 2.5.5  Find the second derivative of \( Y(X) = X^3 + 15X^2 + 3 \).

Technically for this problem, you cannot use the Power Rule since it is taught later in the book.

**Finding the First Derivative:**

We will use two notations to depict derivatives, in case you prefer one over the one.

We have the first derivative:

\[
\frac{dY}{dX} = Y'(X) = \lim_{\Delta X \to 0} \left\{ \frac{Y(X + \Delta X) - Y(X)}{\Delta X} \right\}
\]

\[
= \lim_{\Delta X \to 0} \left\{ \frac{(X + \Delta X)^3 + 15 \cdot (X + \Delta X)^2 + 3}{\Delta X} - \frac{X^3 + 15 \cdot X^2 + 3}{\Delta X} \right\}
\]

Now let’s clean up the numerator. We know that:

\[
\left[ (X + \Delta X)^3 + 15 \cdot (X + \Delta X)^2 + 3 \right] - \left[ X^3 + 15 \cdot X^2 + 3 \right]
\]

\[
= \left[ X^3 + 3 \cdot X^2 \cdot (\Delta X) + 3 \cdot X \cdot (\Delta X)^2 + (\Delta X)^3 + 15 \cdot X^2 + 30 \cdot X \cdot (\Delta X) + 15 \cdot (\Delta X)^2 + 3 \right]
\]

\[
- \left[ X^3 + 15 \cdot X^2 + 3 \right]
\]

\[
= (\Delta X) \cdot \left[ 3 \cdot X^2 + 3 \cdot X \cdot (\Delta X) + (\Delta X)^2 + 30 \cdot X + 15 \cdot (\Delta X) \right]
\]

(2)
Now, let’s plug Equation (2) into Equation (1). Our equation (1) becomes:

\[
\frac{dY}{dX} = Y'(X) = \lim_{\Delta X \to 0} \left\{ \frac{Y(X + \Delta X) - Y(X)}{\Delta X} \right\}
\]

\[
= \lim_{\Delta X \to 0} \left\{ \frac{(X + \Delta X)^3 + 15 \cdot (X + \Delta X)^2 + 3 - X^3 + 15 \cdot X^2 + 3}{\Delta X} \right\}
\]

\[
= \lim_{\Delta X \to 0} \left\{ \frac{(\Delta X) \cdot [3 \cdot X^2 + 3 \cdot X \cdot (\Delta X) + (\Delta X)^2 + 30 \cdot X + 15 \cdot (\Delta X)]}{\Delta X} \right\}
\]

\[
= \lim_{\Delta X \to 0} \left\{ 3 \cdot X^2 + 3 \cdot X \cdot (\Delta X) + (\Delta X)^2 + 30 \cdot X + 15 \cdot (\Delta X) \right\}
\]

\[
= 3 \cdot X^2 + 30 \cdot X
\]

**Finding the Second Derivative:**

We have the second derivative, using Equation (3):

\[
\frac{d^2Y}{dX^2} = Y''(X) = \lim_{\Delta X \to 0} \left\{ \frac{Y'(X + \Delta X) - Y'(X)}{\Delta X} \right\}
\]

\[
= \lim_{\Delta X \to 0} \left\{ \frac{3 \cdot (X + \Delta X)^2 + 30 \cdot (X + \Delta X) - 3 \cdot X^2 - 30 \cdot X}{\Delta X} \right\}
\]

Now let’s clean up the numerator. We know that:

\[
3 \cdot (X + \Delta X)^2 + 30 \cdot (X + \Delta X) - 3 \cdot X^2 - 30 \cdot X
\]

\[
= 3 \cdot [X^2 + 2 \cdot X \cdot (\Delta X) + (\Delta X)^2] + 30 \cdot (X + \Delta X) - 3 \cdot X^2 - 30 \cdot X
\]

\[
= 3 \cdot X^2 + 6 \cdot X \cdot (\Delta X) + 3 \cdot (\Delta X)^2 + 30 \cdot X + 30 \cdot (\Delta X) - 3 \cdot X^2 - 30 \cdot X
\]

\[
= \frac{\Delta X \cdot [6 \cdot X + 3 \cdot (\Delta X) + 30]}{\Delta X}
\]
Now, let’s plug Equation (5) into Equation (4). Our equation (4) becomes:

\[
\frac{d^2Y}{dX^2} = Y''(X) = \lim_{\Delta X \to 0} \left\{ \frac{Y'(X + \Delta X) - Y'(X)}{\Delta X} \right\}
\]

\[
= \lim_{\Delta X \to 0} \left\{ \frac{3 \cdot (X + \Delta X)^2 + 30 \cdot (X + \Delta X) - 3 \cdot X^2 - 30 \cdot X}{\Delta X} \right\}
\]

\[
= \lim_{\Delta X \to 0} \left\{ \frac{(\Delta X)^2 \cdot [6 \cdot X + 3 \cdot (\Delta X) + 30]}{\Delta X} \right\}
\]

\[
= \lim_{\Delta X \to 0} \left[ 6 \cdot X + 3 \cdot (\Delta X) + 30 \right]
\]

\[
\iff \frac{d^2Y}{dX^2} = Y''(X) = 6 \cdot X + 30
\]
0.0.0.1 Problem 2.5.6

Exercise 2.5.6 The growth of cells in a petri dish slows down over time. Is the second derivative of the function giving the number of cells positive or negative?

If we have \( N \equiv \) the number of cells, we are looking at the growth of cells, i.e. \( \frac{dN}{dt} > 0 \)

We now that for second derivative: \( \frac{d^2N}{dt^2} = \frac{d}{dt} \left( \frac{dN}{dt} \right) \)

Since “The growth of cells in a petri dish slows down over time”, we know that \( \frac{dN}{dt} \) decreases over time (i.e. becomes less positive).

Since \( \frac{dN}{dt} \) decreases over time, the rate of change of \( \frac{dN}{dt} \) is negative.

In other words, \( \frac{d^2N}{dt^2} = \frac{d}{dt} \left( \frac{dN}{dt} \right) < 0 \)
Exercise 2.5.7  Why does this make sense?

We can explain this, *algebraically* and *graphically*.

**Using Algebra:**
For $f(X) = c$, we know that regardless of the value of $X$, $f(X)$ will always be equal to $c$. As a result, $f(X + \Delta X) = f(X) = c \iff \Delta f = f(X + \Delta X) - f(X) = c - c = 0$.
Thus,

$$
\frac{d}{dX}(f) = \frac{d}{dX}(c) = \lim_{\Delta x \to 0} \left( \frac{\Delta f}{\Delta x} \right) = \lim_{\Delta x \to 0} \left( \frac{0}{\Delta x} \right) = \lim_{\Delta x \to 0} (0) = 0
$$

**Using Graph:**
If we draw the graph for $f(X) = c$, we will get a horizontal line:

![Graph](image)

To find the derivative of $f(X)$ at any point $X$, we will draw a tangent line to the graph of $f(X)$ at that point $X$. However, since this graph is horizontal, the tangent line is also horizontal as well, regardless of the actual value of $X$. Furthermore, for an horizontal tangent line, we know that its slope is 0.

As a result,

$$
\frac{d}{dX}(f) = \frac{d}{dX}(c) = m = 0
$$
Exercise 2.5.8 Differentiate:

a) \( f(X) = X^5 \)

b) \( f(X) = X^{-3} \)

c) \( f(X) = X^{17.2} \)

Part a: \( f(X) = X^5 \)

We have:

\[
\frac{d}{dX} \left( X^5 \right) = 5 \cdot X^{(5-1)} \iff f'(X) = 5 \cdot X^4
\]

Part b: \( f(X) = X^{-3} \)

We have:

\[
\frac{d}{dX} \left( X^{-3} \right) = (-3) \cdot X^{(-3-1)} \iff f'(X) = (-3) \cdot X^{-4} \iff f'(X) = \frac{-3}{X^4}
\]

Part c: \( f(X) = X^{17.2} \)

We have:

\[
\frac{d}{dX} \left( X^{17.2} \right) = (17.2) \cdot X^{(17.2-1)} \iff f'(X) = (17.2) \cdot X^{16.2}
\]
0.0.0.1 Problem 2.5.9

Exercise 2.5.9 The maximum life-span, \( L \), of a mammalian species increases with average body mass \( B \) as roughly \( L(B) = B^{0.25} \). What is the rate of increase of life-span with body mass?

The rate of increase of life-span with body mass is:

\[
\frac{dL}{dB} = L'(B) = \frac{d}{dB} (B^{0.25}) = (0.25) \cdot B^{(0.25 - 1)} = (0.25) \cdot B^{-0.75}
\]

Another way to write this answer is:

\[
\frac{dL}{dB} = L'(B) = (0.25) \cdot B^{-0.75} = (0.25) \cdot B^{-3/4}
\]

\[
\begin{align*}
\frac{dL}{dB} &= L'(B) = 0.25 \frac{1}{\sqrt[4]{X}}.
\end{align*}
\]
Exercise 2.5.10 Differentiate:

<table>
<thead>
<tr>
<th>Part</th>
<th>Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>a)</td>
<td>$f(X) = 4X^8$</td>
</tr>
<tr>
<td>b)</td>
<td>$f(X) = 3.5X^{-2}$</td>
</tr>
<tr>
<td>c)</td>
<td>$f(X) = \pi X^{4.3}$</td>
</tr>
</tbody>
</table>

**Part a:** $f(X) = 4X^8$

We have:

$$f'(X) = \frac{d}{dX}(4X^8) = 4 \cdot \frac{d}{dX}(X^8) = 4 \cdot (8 \cdot X^{8-1}) \iff f'(X) = 32 \cdot X^7$$

**Part b:** $f(X) = 3.5X^{-2}$

We have:

$$f'(X) = \frac{d}{dX}(3.5X^{-2}) = 3.5 \cdot \frac{d}{dX}(X^{-2}) = 3.5 \cdot (-2)X^{(-2-1)}$$

$$\iff f'(X) = (-7) \cdot X^{-3} \iff f'(X) = \frac{-7}{X^3}$$

**Part c:** $f(X) = \pi X^{4.3}$

We have:

$$f'(X) = \frac{d}{dX}(\pi X^{4.3}) = \pi \cdot \frac{d}{dX}(X^{4.3}) = \pi \cdot (4.3 \cdot X^{(4.3-1)}) \iff f'(X) = 4.3 \cdot \pi \cdot X^{3.3}$$
0.0.0.1 Problem 2.5.11

Exercise 2.5.11 A similar rule holds for subtraction. Why?

To start, let’s recall that:

\[ a - b = a + (-b) = a + (-1) \cdot b \]

Thus, if \( f(X) \) and \( g(X) \) are two functions of \( X \), and we let:

\[ h(X) = f(X) - g(X) = f(X) + (-1) \cdot g(X) \]

Then we have:

\[
\frac{dh}{dX} = \frac{d(f - g)}{dX} = \frac{d(f + (-1) \cdot g)}{dX} = \frac{df}{dX} + \frac{d}{dX} (-1 \cdot g) = \frac{df}{dX} + (-1) \cdot \frac{dg}{dX}
\]

\[\iff \frac{d(f - g)}{dX} = \frac{df}{dX} - \frac{dg}{dX}\]
Exercise 2.5.12 Apply the addition and subtraction rules to calculate the derivative of the function \( f(X) = X + X^2 - 2X^3 + 2X^4 \).

We have:

\[
\frac{df}{dX} = \frac{d}{dX} \left( X + X^2 - 2X^3 + 2X^4 \right)
\]

\[
= \frac{d}{dX}(X) + \frac{d}{dX}(X^2) - \frac{d}{dX}(2X^3) + \frac{d}{dX}(2X^4)
\]

\[
= \left( 1 \cdot X^{1-1} \right) + \left( 2 \cdot X^{2-1} \right) - \left( 2 \cdot 3 \cdot X^{3-1} \right) + \left( 2 \cdot 4 \cdot X^{4-1} \right)
\]

\[
= \left( 1 \cdot X^0 \right) + \left( 2 \cdot X^1 \right) - \left( 6 \cdot X^2 \right) + \left( 8 \cdot X^3 \right)
\]

\[
\iff f'(X) = \frac{df}{dX} = 1 + 2 \cdot X - 6 \cdot X^2 + 8 \cdot X^3
\]
Exercise 2.5.13  What is the rule for differentiating a function of the form $h(X) = f(X) + c$, where $c$ is a constant? Justify your answer in terms of the rules we already know.

We have:

$$\frac{dh}{dX} = \frac{d}{dX} \left( f(X) + c \right) = \frac{d}{dX} \left( f(X) \right) + \frac{d}{dX} \left( c \right) \quad \iff \quad \frac{dh}{dX} = \frac{df}{dX}$$

0 (Constant Rule)
Problem 2.5.14

Exercise 2.5.14 Differentiate the following functions:

a) \( f(t) = \sin(t) \cos(t) \)  

b) \( h(X) = \frac{X^2}{3X + 5} \)

c) \( f(X) = \frac{4X}{\sqrt{X} + 2} \)  

d) \( g(Y) = \left(3Y^6\right) \ln Y \)

Part a: \( f(t) = \sin(t) \cos(t) \)

We have:

\[
\frac{df}{dt} = \frac{d}{dt} \left[ \sin(t) \cos(t) \right] \\
= \cos(t) \cdot \cos(t) + \sin(t) \cdot \left( -\sin(t) \right) \\
\implies \frac{df}{dt} = \cos^2(t) - \sin^2(t)
\]

Other Acceptable Answers include:

\[
\frac{df}{dt} = 2 \cdot \cos^2(t) - 1 \quad \text{OR} \quad \frac{df}{dt} = 1 - 2 \cdot \sin^2(t)
\]

(These two answers are obtained using the property: \( \sin^2(t) + \cos^2(t) = 1 \))

Part b: \( h(X) = \frac{X^2}{3X + 5} \)

We have:

\[
\frac{dh}{dX} = \frac{d}{dX} \left[ \frac{X^2}{3X + 5} \right] \\
= \frac{(3X + 5) \cdot \frac{d}{dX} (X^2) - (X^2) \cdot \frac{d}{dX} (3X + 5)}{(3X + 5)^2} \\
= \frac{(3X + 5) \cdot (2 \cdot X) - (X^2) \cdot (3)}{(3X + 5)^2} \\
\implies \frac{dh}{dX} = \frac{3 \cdot X^2 + 10 \cdot X}{(3X + 5)^2}
\]
Part c: \( f(X) = \frac{4X}{\sqrt{X}+2} \)

Recall that \( \sqrt{X} = X^{1/2} \). With this information, we have:

\[
f'(X) = \frac{df}{dX} = \frac{d}{dX} \left[ \frac{4X}{\sqrt{X}+2} \right]
\]

\[
= \frac{(\sqrt{X}+2) \cdot \frac{d}{dX} (4X) - (4X) \cdot \frac{d}{dX} (\sqrt{X}+2)}{(\sqrt{X}+2)^2}
\]

\[
= \frac{(\sqrt{X}+2) \cdot 4 - (4X) \cdot \left[ \frac{1}{2} \cdot X^{-1/2} \right]}{(\sqrt{X}+2)^2}
\]

\[
= \frac{(\sqrt{X}+2) \cdot 4 - 2 \cdot \sqrt{X}}{(\sqrt{X}+2)^2}
\]

\[
\iff\quad f'(X) = \frac{df}{dX} = \frac{2 \cdot \sqrt{X} + 8}{(\sqrt{X}+2)^2}
\]

Part d: \( g(Y) = (3Y^6) \ln Y \)

We have:

\[
g'(Y) = \frac{dg}{dY} = \frac{d}{dY} \left[ (3Y^6) \ln Y \right]
\]

\[
= \frac{d}{dY} \left[ (3Y^6) \right] \cdot \ln(Y) + (3Y^6) \cdot \frac{d}{dY} \left[ \ln(Y) \right]
\]

\[
= (3 \cdot 6 \cdot Y^5) \cdot \ln(Y) + (3 \cdot Y^6) \cdot \frac{1}{Y}
\]

\[
\iff\quad g'(Y) = \frac{dg}{dY} = 18 \cdot Y^5 \cdot \ln(Y) + 3 \cdot Y^5
\]
Exercise 2.5.15 Write the following expressions of \( h(X) \) as a composition of two functions, one outer function \( f(Y) \) and one inner function \( g(X) \), so that \( f(g(X)) = h(X) \). Then, find the derivative of each.

a) \( h(X) = (X^3 + 1)^2 \)  
b) \( h(X) = \sqrt{X^5} \)  
c) \( h(X) = e^{X^2+1} \)

Part a: \( h(X) = (X^3 + 1)^2 \)

We know that the outer function has the form of \( (\ldots)^2 \). Meanwhile, the inner function takes the form of \( X^3 + 1 \).

Since \( f(\ldots) \) is on the outside, we have:

- \( f(Y) = Y^2 \)
- \( g(X) = X^3 + 1 \)

We know that:

\[
\frac{dh}{dX} = \frac{d(f \circ g)}{dX} = \frac{df}{dg} \cdot \frac{dg}{dX} 
\]

\[
= \left\{ \frac{d}{d(X^3 + 1)}[(X^3 + 1)^2] \right\} \cdot \left\{ \frac{d}{dX}[X^3 + 1] \right\} 
\]

\[
= \left\{ 2 \cdot (X^3 + 1)^{2-1} \right\} \cdot \left\{ 3 \cdot X^{3-1} + 0 \right\} 
\]

\[
= \left\{ 2 \cdot (X^3 + 1) \right\} \cdot \left\{ 3 \cdot X^2 \right\} 
\]

\[
\iff h'(X) = 6 \cdot (X^3 + 1) \cdot X^2 
\]

Part b: \( h(X) = \sqrt{X^5} \)

We know that the outer function has the form of \( \sqrt{\ldots} \). Meanwhile, the inner function takes the form of \( X^5 \).

Since \( f(\ldots) \) is on the outside, we have:

- \( f(Y) = \sqrt{Y} \)
- \( g(X) = X^5 \)
Recall that $\sqrt{X} = X^{1/2}$. With this information, we have:

$$h'(X) = \frac{dh}{dX} = \frac{d(f \circ g)}{dX} = \frac{df}{dg} \cdot \frac{dg}{dX}$$

$$= \left\{ \frac{d}{d(X^5)} \left[ \sqrt{X^5} \right] \right\} \cdot \left\{ \frac{d}{dX} \left[ X^5 \right] \right\}$$

$$= \left\{ \frac{1}{2} \cdot (X^5)^{(1/2)-1} \right\} \cdot \left\{ 5 \cdot X^{5-1} \right\}$$

$$= \left\{ \frac{1}{2} \cdot (X^5)^{-1/2} \right\} \cdot \left\{ 5 \cdot X \right\}$$

$$\iff h'(X) = \frac{5 \cdot X}{2 \cdot \sqrt{X^5}} \iff h'(X) = \frac{5}{2} \cdot \sqrt{X^3}$$

**Part c: $h(X) = e^{X^2+1}$**

We know that the outer function has the form of $e^{(\cdot)}$. Meanwhile, the inner function takes the form of $X^2 + 1$.

Since $f(\ldots)$ is on the outside, we have:

- $f(Y) = e^Y$
- $g(X) = X^2 + 1$

$$h'(X) = \frac{dh}{dX} = \frac{d(f \circ g)}{dX} = \frac{df}{dg} \cdot \frac{dg}{dX}$$

$$= \left\{ \frac{d}{d(X^2 + 1)} \left[ e^{(X^2+1)} \right] \right\} \cdot \left\{ \frac{d}{dX} \left[ X^2 + 1 \right] \right\}$$

$$= \left\{ e^{(X^2+1)} \right\} \cdot \left\{ 2 \cdot X^{2-1} + 0 \right\}$$

$$= \left\{ e^{(X^2+1)} \right\} \cdot \left\{ 2 \cdot X \right\}$$

$$\iff h'(X) = 2 \cdot X \cdot e^{(X^2+1)}$$
**0.0.0.1 Problem 2.5. Further Exercise 1**

**Exercise 2.5.FE 1** Differentiate the following functions:

a) \( f(X) = 2.5X \)

b) \( g(X) = 8X + 4 \)

c) \( f(X) = 3X^4 - 6X^2 + 5X + 10 \)

d) \( \tan(X) = \frac{\sin(X)}{\cos(X)} \)

e) \( y(X) = e^X \sin(X) \)

f) \( f(t) = 2.5 \cos(t + \pi) + 10 \) (Functions like this are often used to model seasonally varying parameters.)

g) \( w(t) = (t^6 + 26t^4 - t^3 + 179)^{73} \)

h) \( f(X) = e^{\sqrt{X}} \)

i) \( f(t) = 3t^7 + 4t^5 - \sqrt{t} \)

j) \( f(X) = \frac{1}{1 + X} \) (You will see this function and the two that follow in more advanced models later in this book.)

k) \( f(X) = \frac{X}{1 + X} \)

l) \( f(X) = \frac{X^2}{1 + X^2} \)

---

**Part a: \( f(X) = 2.5X \)**

\[
f'(X) = \frac{df}{dX} = \frac{d}{dX} (2.5X) = 2.5 \cdot (1) \cdot X^{(1-1)} = 2.5 \cdot 1 \cdot X^0 \iff \frac{df}{dX} = 2.5
\]

**Part b: \( g(X) = 8X + 4 \)**

\[
g'(X) = \frac{dg}{dX} = \frac{d}{dX} (8X + 4) = \frac{d}{dX} (8X) + \frac{d}{dX} (4) = 8 \cdot 1 \cdot X^{(1-1)} \iff \frac{dg}{dX} = 8
\]
**Part c:** \( f(X) = 3X^4 - 6X^2 + 5X + 10 \)

\[
f'(X) = \frac{df}{dX} = \frac{d}{dX}(3X^4 - 6X^2 + 5X + 10)
\]

\[
= \frac{d}{dX}(3X^4) + \frac{d}{dX}(-6X^2) + \frac{d}{dX}(+5X) + \frac{d}{dX}(10)
\]

\[
= \left[(3) \cdot (4) \cdot X^{(4-1)}\right] + \left[(-6) \cdot (2) \cdot X^{(2-1)}\right] + \left[(+5) \cdot X^{(1-1)}\right]
\]

\[
= 12 \cdot X^3 - 12 \cdot X^1 + 5 \cdot X^0
\]

\[
\iff \quad f'(X) = \frac{df}{dX} = 12 \cdot X^3 - 12 \cdot X + 5
\]

**Part d:** \( \tan(X) = \frac{\sin(X)}{\cos(X)} \)

For this question, basically, we are asking what is the derivative for \( f(X) = \tan(X) \). We provide you with the definition of \( \tan(X) \) so you can better apply the derivative rules.

Now, for \( f(X) = \tan(X) \), we can find it’s derivative to be:

\[
f'(X) = \frac{df}{dX} = \frac{d}{dX}\left(\tan(X)\right) = \frac{d}{dX}\left[\frac{\sin(X)}{\cos(X)}\right]
\]

\[
= \frac{\cos(X) \cdot \frac{d}{dX}[\sin(X)] - \sin(X) \cdot \frac{d}{dX}[\cos(X)]}{(\cos(X))^2}
\]

\[
= \frac{\cos(X) \cdot \cos(X) - \sin(X) \cdot (-\sin(X))}{\cos^2(X)}
\]

\[
\iff \quad f'(X) = \frac{d}{dX}\left(\tan(X)\right) = \frac{\cos^2(X) + \sin^2(X)}{\cos^2(X)} = \frac{1}{\cos^2(X)}
\]

\[
\iff \quad f'(X) = \frac{d}{dX}\left(\tan(X)\right) = \sec^2(X)
\]
**Part e:** \( y(X) = e^X \sin(X) \)

\[
y'(X) = \frac{dy}{dX} = \frac{d}{dX} \left[ e^X \sin(X) \right] = \frac{d}{dX} \left[ e^X \right] \cdot \sin(X) + e^X \cdot \frac{d}{dX} \left[ \sin(X) \right]
\]

\[
\iff y'(X) = \frac{dy}{dX} = e^X \cdot \sin(X) + e^X \cdot \cos(X) \iff y'(X) = \frac{dy}{dX} = e^X \cdot \left[ \sin(X) + \cos(X) \right]
\]

**Part f:** \( f(t) = 2.5 \cos(t + \pi) + 10 \) (Functions like this are often used to model seasonally varying parameters.)

In this case we have a composite function:

- The outer function takes the form of: \(2.5 \cdot \cos(...) + 10\)
- The inner function takes the form of: \(t + \pi\)

Thus, with this information, we have the following answer:

\[
f'(t) = \frac{df}{dt} = \frac{d}{dt} \left[ 2.5 \cos(t + \pi) + 10 \right]
\]

\[
= \left\{ \frac{d}{d(t + \pi)} \left[ 2.5 \cos(t + \pi) + 10 \right] \right\} \cdot \left\{ \frac{d}{dt} \left[ t + \pi \right] \right\}
\]

\[
= \left\{ \frac{d}{d(t + \pi)} \left[ 2.5 \cos(t + \pi) \right] + \frac{d}{d(t + \pi)} \left[ 10 \right] \right\} \cdot \left\{ \frac{d}{dt} \left[ t \right] + \frac{d}{dt} \left[ \pi \right] \right\}
\]

\[
= \left\{ 2.5 \cdot \left[ - \sin(t + \pi) \right] \right\} \cdot \left\{ 1 \cdot t^{(1-1)} \right\}
\]

\[
= (-2.5) \cdot \sin(t + \pi) \cdot 1 \cdot \underbrace{t^0}_{1}
\]

\[
\iff f'(t) = \frac{df}{dt} = (-2.5) \cdot \sin(t + \pi) \iff f'(t) = \frac{df}{dt} = (2.5) \cdot \sin(t)
\]

*For those who remember trigonometry, you will recall that: \(\sin(t + \pi) = -\sin(t)\)*
Alternative Method:
For those who remember trigonometry, here is an alternative solution.

\[ f(t) = 2.5 \cdot \cos(t + \pi) + 10 \iff f(t) = -2.5 \cos(t) + 10 \]

Now, the derivative is:

\[ f'(t) = \frac{df}{dt} = \frac{d}{dt}[-2.5 \cos(t) + 10] = \frac{d}{dt}[-2.5 \cos(t)] + \frac{d}{dt}[10] \iff f'(t) = (2.5) \cdot \sin(t) \]

Part g: \( w(t) = \left(t^6 + 26t^4 - t^3 + 179\right)^{73} \)

In this case we have a composite function:

- The outer function takes the form of: \((...)^{73}\)
- The inner function takes the form of: \(t^6 + 26t^4 - t^3 + 179\)

Thus, with this information, we have the following answer:

\[ w'(t) = \frac{dw}{dt} = \frac{d}{dt}\left[\left(t^6 + 26t^4 - t^3 + 179\right)^{73}\right] \]

\[ = \left\{ \frac{d}{d(t^6 + 26t^4 - t^3 + 179)}\left[\left(t^6 + 26t^4 - t^3 + 179\right)^{73}\right]\right\} \cdot \left\{ \frac{d}{dt}\left[t^6\right] + \frac{d}{dt}[26t^4] + \frac{d}{dt}[-t^3] + \frac{d}{dt}[179] \right\} \]

\[ = 73 \cdot \left(t^6 + 26t^4 - t^3 + 179\right)^{72} \cdot \left\{ 6 \cdot t^{(6-1)} + 26 \cdot 4 \cdot t^{(4-1)} + (-1) \cdot 3 \cdot t^{(3-1)} \right\} \]

\[ \iff w'(t) = \frac{dw}{dt} = 73 \cdot \left(t^6 + 26t^4 - t^3 + 179\right)^{72} \cdot \left(6 \cdot t^5 + 104 \cdot t^3 - 3 \cdot t^2\right) \]

\[ \iff w'(t) = \frac{dw}{dt} = 73 \cdot t^2 \cdot \left(t^6 + 26t^4 - t^3 + 179\right)^{72} \cdot \left(6 \cdot t^3 + 104 \cdot t - 3\right) \]
Part h: $f(X) = e^{\sqrt{X}}$

In this case we have a composite function:

- The outer function takes the form of: $e^{(\cdot)}$
- The inner function takes the form of: $\sqrt{X} \equiv X^{(1/2)}$

Thus, with this information, we have the following answer:

$$f'(X) = \frac{df}{dX} = \frac{d}{dX} \left[ e^{\sqrt{X}} \right]$$

$$= \left\{ \frac{d}{d(\sqrt{X})} \left[ e^{\sqrt{X}} \right] \right\} \cdot \left\{ \frac{d}{dX} \left[ \sqrt{X} \right] \right\}$$

$$= \left[ e^{\sqrt{X}} \right] \cdot \left[ \frac{1}{2} \cdot X^{(1/2-1)} \right]$$

$$\iff f'(X) = \frac{df}{dX} = \frac{1}{2} \cdot e^{\sqrt{X}} \cdot X^{(-1/2)}$$

Part i: $f(t) = 3t^7 + 4t^5 - \sqrt{t}$

Recall that: $\sqrt{X} \equiv X^{(1/2)}$. With this information, we have:

$$f'(t) = \frac{df}{dt} = \frac{d}{dt} \left[ 3t^7 + 4t^5 - \sqrt{t} \right]$$

$$= \frac{d}{dt} \left[ 3t^7 \right] + \frac{d}{dt} \left[ 4t^5 \right] + \frac{d}{dt} \left[ -\sqrt{t} \right]$$

$$= \left[ 3 \cdot 7 \cdot t^{(7-1)} \right] + \left[ 4 \cdot 5 \cdot t^{(5-1)} \right] + \left[ (-1) \cdot \frac{1}{2} \cdot t^{(1/2-1)} \right]$$

$$\iff f'(t) = \frac{df}{dt} = 21 \cdot t^6 + 20 \cdot t^4 - \frac{1}{2} \cdot t^{(-1/2)}$$
Part j: \( f(X) = \frac{1}{1+X} \) (You will see this function and the two that follow in more advanced models later in this book.)

**Method #1: Using Quotient Rule**

\[
\frac{df}{dX} = \frac{d}{dX} \left[ \frac{1}{1+X} \right] = \frac{(1+X) \cdot \frac{d}{dX} \left[ 1 \right] - (1) \cdot \frac{d}{dX} \left[ 1 + X \right]}{(1+X)^2} 
\]

\[
\Leftrightarrow \frac{df}{dX} = \frac{-(1) \cdot \frac{d}{dX} \left[ 1 \right] - (1) \cdot \frac{d}{dX} \left[ 1 + X \right]}{(1+X)^2} 
\]

\[
\Leftrightarrow \frac{df}{dX} = \frac{-1}{(1+X)^2} 
\]

**Method #2: Using Chain Rule**

In this case we have a composite function:

- The outer function takes the form of: \( \frac{1}{(...)} \)
- The inner function takes the form of: \( 1 + X \)

Also recall that: \( \frac{1}{Y} = Y^{-1} \). With this information, we have the following answer:

\[
\frac{df}{dX} = \frac{d}{dX} \left[ \frac{1}{1+X} \right] = \frac{d}{d(1+X)} \left[ \frac{1}{1+X} \right] \cdot \left\{ \frac{d}{dX} \left[ 1 + X \right] \right\}
\]

\[
= \left\{ \frac{d}{d(1+X)} \left[ (1+X)^{-1} \right] \right\} \cdot \left\{ \frac{d}{dX} \left[ 1 \right] + \frac{d}{dX} \left[ X \right] \right\}
\]

\[
= (-1) \cdot (1 + X)^{-1} \cdot (0 + 1)
\]

\[
\Leftrightarrow \frac{df}{dX} = \frac{-1}{(1+X)^2} \Leftrightarrow \frac{df}{dX} = \frac{-1}{(1+X)^2} 
\]
Part k: \( f(X) = \frac{X}{1+X} \)

Method #1: Using Quotient Rule

\[
f'(X) = \frac{df}{dX} = \frac{d}{dX} \left[ \frac{X}{1+X} \right] = \frac{(1+X) \cdot \frac{d}{dX} \left[ X \right] - (X) \cdot \frac{d}{dX} \left[ 1+X \right]}{(1+X)^2}
\]

\[\iff f'(X) = \frac{df}{dX} = \frac{(1+X) - (X) \left\{ \frac{d}{dX} \left[ 1 \right] + \frac{d}{dX} \left[ X \right] \right\}}{(1+X)^2} = \frac{(1+X) - (X) \cdot (1)}{(1+X)^2} \]

\[\iff f'(X) = \frac{df}{dX} = \frac{1}{(1+X)^2} \]

Method #2: Using the Answer from Part j

To do this, let’s first manipulate \( f(X) \):

\[
f(X) = \frac{X}{1+X} = 1 + X - \frac{1}{1+X} \]

Thus, we have:

\[
f'(X) = \frac{df}{dX} = \frac{d}{dX} \left[ 1 - \frac{1}{1+X} \right] = \frac{d}{dX} \left[ 1 \right] - \frac{d}{dX} \left[ \frac{1}{1+X} \right] = 0 - \left[ -\frac{1}{(1+X)^2} \right] \text{ (from Part j)}
\]

\[\iff f'(X) = \frac{df}{dX} = \frac{1}{(1+X)^2} \]

Method #3: Using the Answer from Part j + Product Rule

We will rewrite \( f(X) \) in a slightly different way than Method #2:

\[
f(X) = \frac{X}{1+X} = X \cdot \frac{1}{1+X}
\]

Thus, we have:

\[
f'(X) = \frac{df}{dX} = \frac{d}{dX} \left[ X \cdot \frac{1}{1+X} \right] = \frac{d}{dX} \left[ X \right] \cdot \left( \frac{1}{1+X} \right) - \left( X \right) \cdot \frac{d}{dX} \left[ \frac{1}{1+X} \right] \text{ (from Part j)}
\]
\[
\iff f'(X) = \frac{df}{dX} = \frac{1}{1+X} - \left[ \frac{-X}{(1+X)^2} \right] = \frac{1 \cdot (1+X) - X}{(1+X)^2}
\]

\[
\iff f'(X) = \frac{df}{dX} = \frac{1}{(1+X)^2}
\]

**Part l: f(X) = \frac{X^2}{1+X^2}**

**Method #1: Using Quotient Rule**

\[
f'(X) = \frac{df}{dX} = \frac{d}{dX} \left[ \frac{X^2}{1+X^2} \right]
\]

\[
= \frac{(1+X^2) \cdot \frac{d}{dX} \left[ X^2 \right] - (X^2) \cdot \frac{d}{dX} \left[ 1 + X^2 \right]}{(1+X^2)^2}
\]

\[
= \frac{(1+X^2) \cdot \left[ 2 \cdot X^{(2-1)} \right] - (X^2) \cdot \left( 0 + \frac{d}{dX} \left[ X^2 \right] \right)}{(1+X^2)^2}
\]

\[
= \frac{(1+X^2) \cdot \left[ 2 \cdot X \right] - (X^2) \cdot \left( 0 + \left[ 2 \cdot X^{(2-1)} \right] \right)}{(1+X^2)^2}
\]

\[
= \frac{(2 \cdot X) + X^2 \cdot (2 \cdot X) - \left( X^2 \right) \cdot 2 \cdot X}{(1+X^2)^2}
\]

\[
\iff f'(X) = \frac{df}{dX} = \frac{2 \cdot X}{(1+X^2)^2}
\]

**Method #2: Using the Answer from Part k + Chain Rule**

In this case we have a composite function:

- The outer function takes the form of: \( \frac{Y}{1+Y} \)
• The inner function takes the form of: $X^2$

With this information, we have the following answer:

$$f'(X) = \frac{df}{dX} = \frac{d}{dX} \left[ \frac{X^2}{1 + X^2} \right] = \left\{ \frac{d}{d(X^2)} \left[ \frac{X^2}{1 + X^2} \right] \right\} \cdot \left\{ \frac{d}{dX} \left[ X^2 \right] \right\} = \frac{1}{(1 + X^2)^2} \cdot (2 \cdot X)$$

$$\iff f'(X) = \frac{df}{dX} = \frac{2 \cdot X}{(1 + X^2)^2}$$

**Method #3: Using the Answer from Part j + Chain Rule**

To do this, let’s first manipulate $f(X)$:

$$f(X) = \frac{X^2}{1 + X^2} = \frac{1 + X^2 - 1}{1 + X^2} = \frac{1 + X^2}{1 + X^2} - \frac{1}{1 + X^2} = 1 - \frac{1}{1 + X^2}$$

For the later part, we have a composite function:

• The outer function takes the form of: $\frac{1}{1 + Y}$

• The inner function takes the form of: $X^2$

With this information, we have the following answer:

$$f'(X) = \frac{df}{dX} = \frac{d}{dX} \left[ 1 - \frac{1}{1 + X^2} \right] = \frac{d}{dX} \left[ 1 \right] - \frac{d}{dX} \left[ \frac{1}{1 + X^2} \right]$$

$$\iff f'(X) = \frac{df}{dX} = -\left\{ \frac{d}{d(X^2)} \left[ \frac{1}{1 + (X^2)} \right] \right\} \cdot \left\{ \frac{d}{d(X)} \left[ X^2 \right] \right\} = -\left[ \frac{-1}{(1 + X^2)^2} \right] \cdot (2 \cdot X)$$

$$\iff f'(X) = \frac{df}{dX} = \frac{2 \cdot X}{(1 + X^2)^2}$$

**Method #4: Using the Answer from Part j + Chain Rule + Product Rule**

To do this, let’s first manipulate $f(X)$:

$$f(X) = \frac{X^2}{1 + X^2} = X^2 \cdot \frac{1}{1 + X^2}$$

For the later part, we have a composite function:
• The outer function takes the form of: \( \frac{1}{1 + Y} \)
• The inner function takes the form of: \( X^2 \)

With this information, we have the following answer:

\[
\begin{align*}
    f'(X) &= \frac{df}{dX} = \frac{d}{dX} \left[ X^2 \cdot \frac{1}{1 + X^2} \right] \\
    &= \left( \frac{d}{dX} (X^2) \right) \cdot \left[ \frac{1}{1 + X^2} \right] + \left[ X^2 \right] \cdot \left( \frac{d}{dX} \left( \frac{1}{1 + X^2} \right) \right) \\
    &= \left( 2 \cdot X \right) \cdot \left( \frac{1}{1 + X^2} \right) + \left( X^2 \right) \cdot \left\{ \frac{d}{d(X^2)} \left[ \frac{1}{1 + (X^2)} \right] \right\} \cdot \left\{ \frac{d}{d(X)} \left[ X^2 \right] \right\} \\
    &= \frac{2 \cdot X}{1 + X^2} + \left( X^2 \right) \cdot \left( 2 \cdot X \right) \cdot \frac{(-1)}{(1 + X^2)^2} \\
    &= \frac{2 \cdot X \cdot (1 + X^2) - 2 \cdot X^3}{(1 + X^2)^2} = \frac{2 \cdot X + 2 \cdot X^2 - 2 \cdot X^3}{(1 + X^2)^2} \\
\end{align*}
\]

\[ \iff f'(X) = \frac{df}{dX} = \frac{2 \cdot X}{(1 + X^2)^2} \]
0.0.0.1 Problem 2.5. Further Exercise 2

**Exercise 2.5.FE 2** What is the slope of the tangent line to the graph of \( Y = e^{X^2} \) at \( X = 1 \)?

First, let’s find \( \frac{dY}{dX} \). We see that this is a composite function where:

- The outer function takes the form of: \( e^{(\cdot)} \)
- The inner function takes the form of: \( X^2 \)

With this information, we have the following answer:

\[
\frac{dY}{dX} = \frac{d}{dX} \left[ e^{X^2} \right] = \left\{ \frac{d}{d\left(X^2\right)} \left[ e^{X^2} \right] \right\} \cdot \left\{ \frac{d}{dX} \left[ X^2 \right] \right\} = \left\{ e^{X^2} \right\} \cdot \left\{ 2 \cdot X^{(2-1)} \right\} = 2 \cdot X \cdot e^{X^2}
\]

The slope of the tangent line to the graph of \( Y = e^{X^2} \) at \( X = 1 \) is:

\[
m = \left. \frac{dY}{dX} \right|_{X=1} = 2 \cdot 1 \cdot e^{1^2} \iff m = 2 \cdot e \approx 5.436564
\]
0.0.0.1 Problem 2.5. Further Exercise 3

**Exercise 2.5.FE 3**  Find the linear approximation to the function

\[ f(X) = (X + 2)^3 - e^{3X} \]

at \( X_0 = 0 \).

a) First, give your answer in the form \( \Delta f \approx m \cdot \Delta X \).

b) Expand your answer from part (a) by rewriting \( \Delta f \) as \( f(X) - f(X_0) \) and \( \Delta X \) as \( X - X_0 \), and solving for \( f(X) \). (Note: What is \( f(X_0) \)?)

c) What is \( f(0.2) \), approximately?

d) Use your answer from part (b) to write down the equation for the tangent line to \( f(X) \) at \( X_0 = 0 \).

**Part a:** First, give your answer in the form \( \Delta f \approx m \cdot \Delta X \).

We will need to first find an expression for \( m \). The expression for \( m \) is:

\[
m = \frac{df}{dX} = \frac{d}{dX} \left[ (X + 2)^3 - e^{3X} \right] = \frac{d}{dX} \left[ (X + 2)^3 \right] - \frac{d}{dX} \left[ e^{3X} \right]
\]

\[= \left\{ \frac{d}{d(X + 2)} \left[ (X + 2)^3 \right] \cdot \frac{d}{dX} \left[ X + 2 \right] \right\} - \left\{ \frac{d}{d(3X)} \left[ e^{3X} \right] \cdot \frac{d}{dX} \left[ 3X \right] \right\}
\]

\[= \left\{ 3 \cdot (X + 2)^{3-1} \cdot \left[ \frac{d}{dX} \left( X \right) + \frac{d}{dX} \left( 2 \right) \right] \right\} - \left\{ e^{3X} \cdot \left[ 3 \cdot 1 \cdot \frac{X^{(1-1)}}{X^0=1} \right] \right\}
\]

\[= \left\{ 3 \cdot (X + 2)^2 \cdot \left[ \frac{X^{(1-1)}}{X^0=1} \right] \right\} - 3 \cdot e^{3X} = 3 \cdot (X + 2)^2 - 3 \cdot e^{3X}
\]

Now, at \( X_0 = 0 \), our slope becomes:

\[
m = \left. \frac{df}{dX} \right|_{X=0} = 3 \cdot (0 + 2)^2 - 3 \cdot e^{3 \cdot 0} = 3 \cdot 2^2 - 3 \cdot e^0 = 3 \cdot 4 - 3 \cdot 1 = 9
\]

Thus, our linear approximation to the function at \( X = X_0 = 0 \) is:

\[\Delta f \approx m \cdot \Delta X \iff \Delta f \approx \left. \frac{df}{dX} \right|_{X=0} \cdot \Delta X \]

\[\iff \Delta f \approx 9 \cdot \Delta X \]  \( \text{Eq. (1)} \)
Part b: Expand your answer from part (a) by rewriting $\Delta f$ as $f(X) - f(X_0)$ and $\Delta X$ as $X - X_0$, and solving for $f(X)$. (Note: What is $f(X_0)$?)

For $X = X_0 = 0$, we have:

$$f(X_0) = (0 + 2)^3 - e^{3 \cdot 0} = 2^3 - e^0 = 8 - 1 = 7$$

Thus, our equation from part (a) (i.e. Equation (1)) becomes:

$$\Delta f \approx 9 \cdot \Delta X \iff f(X) - f(X_0) \approx 9 \cdot (X - X_0) \iff f(X) - 7 \approx 9 \cdot (X - 0)$$

$$\iff f(X) \approx 9 \cdot X + 7$$ (2)

Part c: What is $f(0.2)$, approximately?

Using Equation (2), we have:

$$f(0.2) \approx 9 \cdot 0.2 + 7 \iff f(0.2) \approx 8.8$$

Part d: Use your answer from part (b) to write down the equation for the tangent line to $f(X)$ at $X_0 = 0$.

From part (a), we have the linear approximation to the function $f(X)$ at $X_0 = 0$ (i.e. Equation (1)) to be:

$$f(X) \approx 9 \cdot X + 7$$

The tangent line to the function $f(X)$ at any point $X = X_0$ is the linear approximation to the function $f(X)$ at that point $X = X_0$. Thus, the equation for the tangent line to $f(X)$ at $X_0 = 0$ will have the exact same form as Equation (1), i.e.:

$$Y = 9 \cdot X + 7$$
Problem 2.5. Further Exercise 4

Exercise 2.5.FE 4 In mammals, resting metabolic rate $M$ is related to body mass $B$ as approximately

\[ M = 0.8B^{3/4} \]

a) Find the linear approximation to this function for a body mass of 100 grams.

b) An animal species that currently averages 100 grams in mass evolves to have an average mass of 110 grams. Use the linear approximation to estimate how much its metabolic rate would change.

Part a: Find the linear approximation to this function for a body mass of 100 grams.

For our solution, we will go after the format: \( \Delta M \approx m \cdot \Delta B \)

First, let’s find the expression for \( m \) at \( B = 100 \). In general, we have:

\[ m = \frac{dM}{dB} = \frac{d}{dB} \left[ 0.8 \cdot B^{3/4} \right] = 0.8 \cdot \frac{3}{4} \cdot B^{(3/4-1)} = 0.6 \cdot B^{(-1/4)} = \frac{0.6}{\sqrt[4]{B}} \]

Thus, at \( B = 100 \), we have:

\[ m = \frac{0.6}{\sqrt[4]{100}} = \frac{6}{10\sqrt{10}} = \frac{3\sqrt{10}}{50} \]

As a result, the linear approximation to this function for a body mass of 100 grams is:

\[ \Delta M \approx \frac{3\sqrt{10}}{50} \cdot \Delta B \iff \Delta M \approx 0.18974 \cdot \Delta B \]

Part b: An animal species that currently averages 100 grams in mass evolves to have an average mass of 110 grams. Use the linear approximation to estimate how much its metabolic rate would change.

From part (a), we have the equation: \( \Delta M \approx \frac{3\sqrt{10}}{50} \cdot \Delta B \)

As a result, if the animal species evolves to have an average mass of 110 grams (from 100 grams), its metabolic rate would change by:

\[ \Delta M \approx \frac{3\sqrt{10}}{50} \cdot \Delta B \iff \Delta M \approx \frac{3\sqrt{10}}{50} \cdot (110 - 100) \iff \Delta M \approx \frac{3\sqrt{10}}{5} \iff \Delta M \approx 1.8974 \]

In other words, its metabolic rate would **INCREASE** by \( \frac{3\sqrt{10}}{5} \approx 1.8974 \).
Exercise 2.6.1 Why is the +c necessary? Find \( \frac{d}{dt}(t^2 + 5) \) and \( \frac{d}{dt}(t^2 - 1) \).

**Part 1: Why is the +c necessary?**

Ideally, when we find an antiderivative to a function, let’s say \( f'(x) \), we will recover every single possible part of \( f(x) \), for which:

\[
\frac{d}{dx}[f(x)] = f'(x)
\]

Now, the tricky part is if \( f(x) \) contains a constant term, +c. We know that for \( f(x) = (...) + c \),

\[
\frac{d}{dx}[f(x)] = \frac{d}{dx}[(...) + c] = \frac{d}{dx}[(...) + c] = \frac{d}{dx}[(...)] + 0 = \frac{d}{dx}[(...)]
\]

As we can see, the Observed \( f'(x) \) often omits the +0 component that might be a part of the Actual \( f'(x) \). Thus, if we try to find the antiderivative of only the Observed \( f'(x) \), we are going to leave out the +0 part that may give rise to the +c component, that was originally a part of \( f(x) \) (i.e. we only recover the (...) part, but without the +c part).

**Part 2: Find \( \frac{d}{dt}(t^2 + 5) \)**

\[
\frac{d}{dt}[t^2 + 5] = \frac{d}{dt}[t^2] + \frac{d}{dt}[5] = 2 \cdot t^{(2-1)} \iff \frac{d}{dt}[t^2 + 5] = 2 \cdot t
\]

**Part 3: Find \( \frac{d}{dt}(t^2 - 1) \)**

\[
\frac{d}{dt}[t^2 - 1] = \frac{d}{dt}[t^2] + \frac{d}{dt}[-1] = 2 \cdot t^{(2-1)} \iff \frac{d}{dt}[t^2 - 1] = 2 \cdot t
\]
Problem 2.6.2

Exercise 2.6.2 Find the antiderivative of $f'(X) = 3X^2$.

In this case, since $f'(X)$ is in the form of $(... \cdot X^{(\cdot)})$ (i.e. the form of a Power Function), we will be looking at the Power Rule (Page 94). We know that for power functions, for any constant $n \neq 0$,

$$\frac{d}{dX} (X^n) = n \cdot X^{n-1}$$

Now if we combine this with the Constant Multiple Rule, we have, for any constants $k, n \neq 0$:

$$\frac{d}{dX} (k \cdot X^n) = k \cdot n \cdot X^{n-1}$$

We are in search of a function $f(X) = k \cdot X^n$ for which:

$$\frac{d}{dX} (k \cdot X^n) = k \cdot n \cdot X^{n-1} = 3 \cdot X^2$$

$$\iff \begin{cases} n - 1 = 2 \iff n = 3 \\ k \cdot n = 3 \iff k = 1 \end{cases}$$

Thus, we see that $f(X)$ will take the form of: $f(X) = 1 \cdot X^3 = X^3$

Finally, don’t forget that we need the “+c” part, as discussed in Exercise 2.6.1. As a result, the antiderivative of $f'(X) = 3X^2$ is:

$$f(X) = X^3 + c$$
0.0.0.1 Problem 2.6.3

**Exercise 2.6.3** In this table, what are the red entries in the last column? The black entries?

Recall that:

\[
\frac{\Delta X}{\Delta t} \approx \frac{dX}{dt} \iff \frac{\Delta X}{\Delta t} \approx X'(t) \iff \Delta X \approx X'(t) \cdot \Delta t
\]

We know that the black entries represent the \(X_{\text{old}}\) and \(X_{\text{new}}\) values in each step. Then, the red entries represent the \(\Delta X\) that we obtain from the respective steps.

In other words, we have:

\[
X_1 = X_0 + \Delta X = X_0 + X'(X_0) \cdot \Delta t
\]
\[
X_2 = X_1 + \Delta X = X_1 + X'(X_1) \cdot \Delta t
\]
\[
X_3 = X_2 + \Delta X = X_2 + X'(X_2) \cdot \Delta t
\]

[...]
\[
X_{n+1} = X_n + \Delta X = X_n + X'(X_n) \cdot \Delta t
\]
Exercise 2.6.4  Compute:

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a) [ \sum_{k=0}^{3} 2k ]</td>
<td>b) [ \sum_{k=0}^{4} k^3 ]</td>
<td>c) [ \sum_{k=0}^{3} 6k + 2 ]</td>
</tr>
</tbody>
</table>

Part a: \[ \sum_{k=0}^{3} 2k \]

We have:

\[
\sum_{k=0}^{3} 2k = \left[ 2 \cdot (k = 0) \right] + \left[ 2 \cdot (k = 1) \right] + \left[ 2 \cdot (k = 2) \right] + \left[ 2 \cdot (k = 3) \right]
\]

\[
\iff \sum_{k=0}^{3} 2k = (2 \cdot 0) + (2 \cdot 1) + (2 \cdot 2) + (2 \cdot 3) = 0 + 2 + 4 + 6
\]

\[
\iff \sum_{k=0}^{3} 2k = 12
\]
Part b: \( \sum_{k=0}^{4} k^3 \)

We have:

\[
\sum_{k=0}^{4} k^3 = [(k = 0)^3] + [(k = 1)^3] + [(k = 2)^3] + [(k = 3)^3] + [(k = 4)^3]
\]

\[
\iff \sum_{k=0}^{4} k^3 = \left( 0^3 \right) + \left( 1^3 \right) + \left( 2^3 \right) + \left( 3^3 \right) + \left( 4^3 \right) = 0 + 1 + 8 + 27 + 64
\]

\[
\iff \sum_{k=0}^{4} k^3 = 100
\]

Part c: \( \sum_{k=0}^{3} 6k + 2 \)

We have:

\[
\sum_{k=0}^{3} 6k + 2 = \left( 6 \cdot (k = 0) + 2 \right) + \left( 6 \cdot (k = 1) + 2 \right) + \left( 6 \cdot (k = 2) + 2 \right) + \left( 6 \cdot (k = 3) + 2 \right)
\]

\[
\iff \sum_{k=0}^{3} 6k + 2 = \left( 6 \cdot 0 + 2 \right) + \left( 6 \cdot 1 + 2 \right) + \left( 6 \cdot 2 + 2 \right) + \left( 6 \cdot 3 + 2 \right) = 2 + 8 + 14 + 20
\]

\[
\iff \sum_{k=0}^{3} 6k + 2 = 44
\]
Exercise 2.6.5  Find the Riemann sum for \( f(X) = X^2 + 5 \) between \( X = 0 \) and \( X = 2 \) using a step size of 0.5.

Let’s draw the graph to get an idea of what we need to do for this Riemann sum:

As you can see, we will find the areas of each of those yellow rectangles, and add all of them up. Basically, the width of each rectangle is \( \Delta X = 0.5 \). The height varies, more specifically the height of each rectangle is exactly equal to the \( y \)-coordinate of the red point situated on the upper left corner of that specific rectangle. Now let’s look at the values of those points. We will stretch the graph horizontally to fit in the characters...
It is important to note that as we draw the rectangles, we start with $X = 0$ and end at $X = 1.5$, and not $X = 2$ (because otherwise, we will reach $X = 2.5$, which is not what we want).

Our Riemann Sum for $f(X) = X^2 + 5$ between $X = 0$ and $X = 2$ using a step size of 0.5 is:

$$
\sum \left[ \text{Areas of the Rectangle} \right] = \sum \left[ \left( \frac{\text{Height}}{f(X)} \right) \cdot \left( \frac{\text{Width}}{\Delta X} \right) \right] \\
= f(0) \cdot 0.5 + f(0.5) \cdot 0.5 + f(1) \cdot 0.5 + f(1.5) \cdot 0.5 \\
= (0^2 + 5) \cdot 0.5 + (0.5^2 + 5) \cdot 0.5 + (1^2 + 5) \cdot 0.5 + (1.5^2 + 5) \cdot 0.5 \\
\iff \sum \left[ \text{Areas of the Rectangle} \right] = 11.75
$$

**Potential Common Error:**

Some of you might feel inclined to write:

$$
\sum \left[ \text{Areas of the Rectangle} \right] = \sum_{x=0}^{x=1.5} \left[ f(x) \cdot 0.5 \right]
$$

You might think that this is correct because you go from $X = 0$ and end at $X = 1.5$. However, this is **INCORRECT**, because the summation only goes by **INTEGERS**. Hence,

$$
\sum_{x=0}^{x=1.5} \left[ f(x) \cdot 0.5 \right] = (0^2 + 5) \cdot 0.5 + (1^2 + 5) \cdot 0.5 = 5.125 \neq 11.75
$$

In this case, you miss out on both $X = 0.5$ and $X = 1.5$.

To use the summation notation, you first have to convert the series $(0.0, 0.5, 1.0, 1.5)$ into a series that involves taking steps in integers values. For example, we can have $(0.5 \ast 0, 0.5 \ast 1, 0.5 \ast 2, 0.5 \ast 3)$. The two series have same numerical components, but you can use the second one with the summation operation. Then, our notation becomes:

$$
\sum \left[ \text{Areas of the Rectangle} \right] = \sum_{k=0}^{k=3} \left[ f(0.5 \ast k) \cdot 0.5 \right] = \sum_{k=0}^{k=3} \left\{ \left( (0.5 \ast k)^2 + 5 \right) \cdot 0.5 \right\} \\
\iff \sum \left[ \text{Areas of the Rectangle} \right] = 11.75
$$
0.0.0.1 Problem 2.6.6

**Exercise 2.6.6** Find the distance traveled by the car in 8 seconds if \( V(t) = 4\sqrt{500 - t^3} \). Use a step size of 0.5. You may want to use SageMath or a spreadsheet to help with the calculation.

Since we are going from \( t = 0 \) to \( t = 8 \) in step size of 0.5, the last value of \( t \) at which we will evaluate \( V(t) \) is \( t = 7.5 \) (because if we evaluate at \( t = 8 \), then we are basically calculating the distance travelled in 8.5 seconds).

<table>
<thead>
<tr>
<th>( t )</th>
<th>( V(t) = 4\sqrt{500 - t^3} )</th>
<th>( V(t) \cdot \Delta t )</th>
<th>( X(t) = \sum V(t) \cdot \Delta t )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>-</td>
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</tr>
<tr>
<td>7.5</td>
<td>35.35534</td>
<td>17.67767</td>
<td>617.25156</td>
</tr>
</tbody>
</table>

Thus, the total distance traveled by the car in 8 seconds is: **617.25156 ft**

*For those who want to use the summation notation*, here is the answer for you (but we will let you work through the math):

\[
\text{Distance Traveled} = \sum_{k=0}^{k=15} \left[ V(0.5 \cdot k) \cdot 0.5 \right] = \sum_{k=0}^{k=15} \left[ 4\sqrt{500 - (0.5 \cdot k)^3} \cdot 0.5 \right] = 617.25156
\]
0.0.0.1 Problem 2.6.7

Exercise 2.6.7 According to CDC data, the average American six-year-old girl weighs 42.5 pounds, and the average ten-year-old girl weighs 75 pounds. What is the area under the growth (rate of change of weight) function between \( t = 6 \) and \( t = 10 \)?

In this case, we have:

- \( f(t) \equiv \) the growth (rate of change of weight) at time \( t \)
- \( F(t) \equiv \) the weight at time \( t \)

And we know that: \( F'(t) = f(t) \)

The area under the growth (rate of change of weight) function between \( t = 6 \) and \( t = 10 \) is:

\[
\int_{6}^{10} f(t) \cdot dt = F(10) - F(6) = 75 - 42.5 \iff \int_{6}^{10} f(t) \cdot dt = 32.5
\]