1. Let $S$ be a subset of a topological space $X$ and $(s_n)$ a sequence in $S$. Prove that $(s_n)$ converges to $x_0 \in S$ if and only if $(s_n)$ converges to $x_0$ as a sequence in $X$.

Suppose $(s_n)$ converges to $x_0$ in $S$. Given $U$ open in $X$ containing $x_0$, there exists $N$ such that $n > N$ implies $s_n \in U \cap S$ so $s_n \in U$, hence $(s_n)$ converges to $x_0$ in $X$.

Suppose $(s_n)$ converges to $x_0$ in $X$. Given $V$ open in $S$ containing $x_0$, then $V = S \cap U$ for some $U$ open in $X$. Therefore there exists $N$ such that $n > N$ implies $s_n \in U$.

But $s_n \in S$ so $s_n \in S \cap U = V$ and $(s_n)$ converges to $x_0$ in $S$. 
2. Let $S$ be a subset of $X$. (a) Define $\partial S$, the boundary of $S$. (b) Prove that $S$ is an open subset of $X$ if and only if $S \cap \partial S = \emptyset$.

(a) $x_0 \in \partial S$ if, given $U$ open in $X$ containing $x_0$, then $U \cap S \neq \emptyset$ and $U \cap (X \setminus S) \neq \emptyset$.

(b) If $S$ is open in $X$ then if $x_0 \in S$ there exists $U$ open in $X$ such that $U \subseteq S$ so $U \cap (X \setminus S) = \emptyset$ and if $x_0 \notin \partial S$ so $x_0 \notin \partial S = \emptyset$. If $S \cap \partial S = \emptyset$ then $x_0 \in S$ implies $X \setminus S$ so there exists $U$ open in $X$ containing $x_0$ such that $U \cap (X \setminus S) = \emptyset$ and thus $x_0 \in S$ so $S$ is open.
3. Let \( f_n : X \to Y \) be functions from a topological space \( X \) to a metric space \( (Y, d) \). (a) Define: \((f_n)\) converges uniformly to \( f : X \to Y \). (b) Prove that if \((f_n)\) converges uniformly to \( f \) and all the \( f_n \) are continuous, then \( f \) is also continuous.

(a) Given \( \varepsilon > 0 \) there exists \( N \) such that
\[ n > N \implies d (f_n(x), f(x)) < \varepsilon \quad \text{for all} \quad x \in X. \]

(b) Let \( x_0 \in X \) and given \( \varepsilon > 0 \) there exists \( N \) such that \( n \geq N \) implies
\[ d (f_n(x_0), f(x_0)) < \varepsilon/3. \]
In particular, \( d (f_n(x_0), f(x_0)) < \varepsilon/3 \). Since \( f_n \) is continuous at \( x_0 \), there exists \( U \) open in \( X \) containing \( x_0 \) such that \( x \in U \) implies
\[ d (f_n(x), f_N(x)) < \varepsilon/3. \]
Therefore if \( x \in U \) then
\[ d (f(x), f(x_0)) \leq d (f(x), f_N(x)) + d (f_N(x), f_N(x_0)) + d (f_N(x_0), f(x_0)) < \varepsilon \]
so \( f \) is continuous at \( x_0 \).
4. Let $S$ be a compact subset of a Hausdorff space $X$ and let $x_0 \in X \setminus S$. Prove there exist disjoint open sets $U, V$ in $X$ such that $x_0 \in U$ and $S \subseteq V$.

Since $X$ is Hausdorff, for each $s \in S$ there exist $U(s), V(s)$ open in $X$ such that $x_0 \in U(s)$, $s \in V(s)$ and $U(s) \cap V(s) = \emptyset$. Then $\{V(s)\}_{s \in S}$ is an open cover of the compact set $S$ so there is a finite subcover $\{V(s_j)\}_{j=1}^n$. Let $V = \bigcup_{j=1}^n V(s_j)$ and $U = \bigcap_{j=1}^n U(s_j)$. Then $U$ and $V$ are open in $X$, $S \subseteq V$ and $x_0 \in U$ and $U \cap V = \emptyset$. 
5. Let \( f: X \to Y \) be a continuous function such that \( f(X) = Y \). Prove that if \( X \) is a separable space then \( Y \) is also separable.

Let \( D \) be a dense countable subset of \( X \), then \( f(D) \) is a countable subset of \( Y \).

Let \( U \) be an open subset of \( Y \). Since \( f \) is continuous, \( f^{-1}(U) \) is open in \( X \) and since \( f \) is onto, \( f^{-1}(U) \) is nonempty so there exists \( x_0 \in f^{-1}(U) \cap D \). Therefore \( f(x_0) \in U \cap f(D) \) so \( f(D) \) is dense in \( Y \).