1. (a) Let $f: X \to Y$ be a function where $X$ and $Y$ are topological spaces. Define: $f$ is continuous. (b) Let $f: X \to Y$ and $g: Y \to Z$ be continuous functions. Prove that $gf: X \to Z$ is continuous.

[15 points] (a) If $U$ is an open subset of $Y$, then $f^{-1}(U)$ is an open subset of $X$.

[15 points] (b) Let $U$ be an open subset of $Z$. Since $g$ is continuous, then $g^{-1}(U)$ is an open subset of $Y$. Since $f$ is continuous, then $f^{-1}(g^{-1}(U)) = (gf)^{-1}(U)$ is open in $X$ so $gf$ is continuous.
2. Prove that a compact space has the finite intersection property, that is, if \( \{E_\alpha\}_{\alpha \in A} \) is a family of closed subsets of \( X \) such that \( \bigcap_{j=1}^{m} E_{\alpha_j} \neq \emptyset \) for every finite subfamily \( \{E_{\alpha_1}, \ldots, E_{\alpha_m}\} \), then \( \bigcap_{\alpha \in A} E_{\alpha} \neq \emptyset \).

To prove the contrapositive, suppose \( \bigcap_{\alpha \in A} E_{\alpha} = \emptyset \). Then \( X \setminus \bigcap_{\alpha \in A} E_{\alpha} = \bigcup_{\alpha \in A} (X \setminus E_{\alpha}) = X \).

So \( \{X \setminus E_{\alpha}\}_{\alpha \in A} \) is an open cover of \( X \).

Since \( X \) is compact, there is a finite subcover \( \{X \setminus E_{\alpha_j}\}_{j=1}^{n} \). Therefore

\[
X = \bigcup_{j=1}^{n} (X \setminus E_{\alpha_j}) = X \setminus \bigcap_{j=1}^{n} E_{\alpha_j} \text{ and thus } \bigcap_{j=1}^{n} E_{\alpha_j} = \emptyset .
\]
3. Let \( S \) be a subset of a topological space \( X \). (a) Define \( \bar{S} \), the closure of \( S \). (b) Prove that the complement of the closure of \( S \) is the interior of the complement of \( S \), that is, \( X \setminus \bar{S} = \text{int}(X \setminus S) \).

5 points] (a) \( x \in \bar{S} \) if and only if given an open set \( U \) containing \( x \), then \( U \cap S \neq \emptyset \).

15 points] (b) Let \( x \in \text{int}(X \setminus S) \) then \( x \in U \subseteq X \setminus S \) where \( U \) is open in \( X \). Since \( U \cap S = \emptyset \), then \( x \notin \overline{S} \) so \( \text{int}(X \setminus S) = X \setminus \overline{S} \).

If \( x \in X \setminus \overline{S} \) then there exists \( U \) open in \( X \) containing \( x \) such that \( U \cap S = \emptyset \). Therefore \( x \in \text{int}(X \setminus S) \) and \( X \setminus \overline{S} \subseteq \text{int}(X \setminus S) \).
4. (a) State and (b) prove Lindelof's Theorem for topological spaces.

[5 points] (a) Every open cover of a second-countable topological space has a countable subcover.

[15 points] (b) Let \( B = \{ V_j \}_{j=1}^{\infty} \) be a countable base for the second-countable space \( X \). Let \( \{ U_a \}_{a \in A} \) be an open cover of \( X \). For \( x \in X \), \( x \in U_a \) for some \( a \in A \). Since \( B \) is a base then \( x \in V_j \subseteq U_a \) for some \( j \). Let \( C \subseteq B \) be those \( V_j \) such that \( V_j \subseteq U_a \) for some \( a \in A \), then \( \{ V_j \}_{j \in C} \) is an open cover of \( X \). For each \( j \) choose \( U_{a_j} \) such that \( V_j \subseteq U_{a_j} \) then \( \{ U_{a_j} \}_{j=1}^{\infty} \) is a countable subcover of \( \{ U_a \}_{a \in A} \).
5. A subset $S$ of a topological space $X$ is a retract of $X$ if there is a continuous function $r: X \to S$ such that $r(s) = s$ for all $s \in S$. Prove that if $X$ is a Hausdorff space and $S$ is a retract of $X$, then $S$ is a closed subset of $X$. (Hint: Prove that $X \setminus S$ is open.)

To prove that $X \setminus S$ is open, let $x_0 \in X \setminus S$. Since $r(x_0) \in S$ then $r(x_0) \neq x_0$.

So since $X$ is Hausdorff there exist open sets $U, V$ such that $x_0 \in U$, $r(x_0) \in V$ and $U \cap V = \emptyset$. Since $r$ is continuous, $r^{-1}(V)$ is open in $X$. Let $W = r^{-1}(V) \cap U$ which is open and contains $x_0$. Then $W \cap V = \emptyset$ since $U \cap V = \emptyset$. Thus if $x \in W$ then $x \in V$ it follows that $r(x) \neq x$ so $x \notin S$. We conclude that $W \subseteq X \setminus S$ and therefore $X \setminus S$ is open.