Equilibria in Two Dimensions

In the previous lab, you used Sage to classify the equilibria of one-variable dynamical systems according to their stability. The idea of equilibria being stable or unstable continues to apply in two (or more) dimensions but develops some extra twists due to the possibility of rotation.

In one dimension, a point representing the state of a system can only move left or right and a change from moving left to moving right requires the point to pass through an equilibrium. Since no change occurs at an equilibrium, the system just stays there for all time and never changes direction. On the other hand, you have been seeing oscillations in two- and higher-dimensional systems since the earliest days of this class. In this lab, you will use Sage to explore the different types of behavior possible in two-variable systems.

Romeo and Juliet, Daytime TV Edition

The system we will study is the Romeo and Juliet model that you have seen before. In this model, Romeo and Juliet are in a relationship. \( R \) represents Romeo’s love (or, if negative, hate) for Juliet and \( J \) represents Juliet’s love or hate for Romeo. Each person can respond both to the other person’s emotions and to their own. The differential equations for this model are \( R' = aR + bJ \) and \( J' = cR + dJ \), where all the parameters can be either positive or negative. This simple model exhibits all the types of equilibria possible in two dimensions, for reasons you will study in LS 30B. We will now use vector fields, trajectories and time series to examine these equilibria and the behavior associated with them.

Getting started

To start out, do the following.

**Exercise 1.** By hand, find the equilibria of the system \( R' = aR + bJ \), \( J' = cR + dJ \).

**Exercise 2.** Declare \( a, b, c, d, R \) and \( J \) as symbolic variables.

**Exercise 3.** Make a list of eight points, two in each quadrant of the \( R-J \) plane, to serve as initial conditions for simulations. Recall that points are defined using parentheses: \((R1, J1)\) is a point.
We can now investigate the different ways Romeo and Juliet can interact and how this affects the dynamics of their relationship.

**Ignoring themselves, ignoring each other**

In the simplest Romeo-Juliet model, each person responds only to their own feelings, ignoring those of the other. In this case, the equations have the form $R' = aR$, $J' = dJ$. There are several ways in which this can play out, depending on the signs of $a$ and $d$.

**Exercise 4.** Pick a positive value for $a$ and $d$. (For simplicity, make them equal to each other.) Plot the system’s vector field, letting both $R$ and $J$ range from -10 to 10. Is the equilibrium stable or unstable?

**Exercise 5.** Now, make $a$ and $d$ negative and plot the vector field. Is the equilibrium stable or unstable?

**Exercise 6.** Interpret the results of the previous two exercises in terms of the relationship between Romeo and Juliet. HINT: If you have trouble doing this from just the vector field, try running some simulations for various initial conditions using `desolve_odeint`.

The equilibria you saw in the previous problems are pretty straightforward equivalents of stable and unstable equilibria in one dimension. However, a slight change can cause a surprise to emerge.

**Exercise 7.** Keeping the absolute value of $a$ and $d$ the same, make one of them positive and the other negative. Plot the vector field as before. Does the equilibrium appear stable or unstable?

**Exercise 8.** Overlay a large red point at the equilibrium on top of the vector field (use the `point` command to plot points) and assign this plot to a variable. Then, use a for loop to simulate the dynamics of the system for each initial condition in the list you defined earlier, make a plot of the resulting trajectory and add it to the existing vector field. View the result. HINT: If your trajectories go outside the vector field, display the plot using `show` and specify `xmin`, `xmax`, `ymin` and `ymax`, as necessary.

**Exercise 9.** Describe what you see in purely visual and dynamical terms, without referring to the meaning of the model. Is the equilibrium point stable or unstable?

**Exercise 10.** Run simulations and plot time series for a couple of different initial conditions. Draw on the vector field, trajectories and time series to describe what is happening to Romeo and Juliet’s relationship in this model.
This kind of equilibrium point is called a *saddle point*. It can only occur in two or more dimensions.

Perhaps the most intuitive Romeo-Juliet model is the one in which each person responds only to the other’s feelings: $R' = bJ$ and $J' = cR$.

**Exercise 11.** Pick a positive value for $b$ and $c$, keeping them equal to each other. Plot the vector field. What kind of equilibrium point appears?

**Exercise 12.** What happens if $b$ and $c$ are both negative?

As before, something new appears when the parameters have different signs.

**Exercise 13.** Keeping the absolute value of $b$ and $c$ the same, make one of them positive and the other negative. Plot the vector field as before.

**Exercise 14.** Overlay a large red point at the equilibrium on top of the vector field (use the `point` command to plot points) and assign this plot to a variable. Then, use a for loop to simulate the dynamics of the system for each initial condition in the list you defined earlier, make a plot of the resulting trajectory and add it to the existing vector field. View the result.

**Exercise 15.** Describe what you see in purely visual and dynamical terms, without referring to the meaning of the model. Is the equilibrium point stable or unstable?

**Exercise 16.** Run simulations and plot time series for a couple of different initial conditions. Draw on the vector field, trajectories and time series to describe what is happening to Romeo and Juliet’s relationship in this model.

In this model, the equilibrium point only provides a point for trajectories to orbit. This kind of equilibrium is called a *center*.

**Romeo and Juliet interact**

In the previous models, Romeo and Juliet either responded only to themselves or only to the other person. When they respond to both themselves and the other person, we can get the same equilibria as before but also see some new ones.

Suppose Romeo is excited by Juliet’s love but unaffected by his own feelings. However, his love for Juliet makes her dislike him but his dislike for her makes her like him. To make things even more interesting, let’s say Juliet doesn’t like being in love, so her own emotions cause themselves to die down, but this effect is weak. This might give us the equations $R' = J$ and $J' = -R - 0.05J$. 
**Exercise 17.** Plot the vector field for this system. Since it will be hard to tell exactly what’s going on from the vector field alone, pick an initial condition, run a simulation and overlay the trajectory on top of the vector field.

**Exercise 18.** Plot time series for $R$ and $J$. Describe what is happening in terms of Romeo and Juliet’s relationship.

**Exercise 19.** This kind of equilibrium is called a stable spiral. Why is the word “stable” appropriate?

We can see other interesting dynamics if Juliet is a little bit excited by her own emotions. Then the equations are $R' = J$ and $J' = -R + 0.05J$.

**Exercise 20.** Plot the vector field for this system and overlay a trajectory on top of it.

**Exercise 21.** Plot time series for $R$ and $J$. Describe what is happening in terms of Romeo and Juliet’s relationship.

**Exercise 22.** What might this kind of equilibrium be called?