1. Define $I: \mathbb{R} \setminus \{0\} \to \mathbb{R}$ by $I(x) = 1/x$. Prove directly from the definition of continuity that $I$ is a continuous function. That is, for this problem do not refer to any theorems from the course.
2. A sequence of functions \( (f_n: X \to \mathbb{R}) \) is *uniformly bounded* if there exists \( M(f) > 0 \) such that \( |f_n(x)| < M(f) \) for all \( x \in X \) and all \( n \in \mathbb{N} \). Suppose \( (f_n, g_n: X \to \mathbb{R}) \) are uniformly bounded sequences, that \( (f_n) \) converges to \( f \) uniformly and \( (g_n) \) converges to \( g \) uniformly. Prove that \( (f_n g_n: X \to \mathbb{R}) \) defined by \( (f_n g_n)(x) = f_n(x)g_n(x) \) converges uniformly to \( fg: X \to \mathbb{R} \) where \( (fg)(x) = f(x)g(x) \).
3. Prove that a subset $K$ of a Euclidean space is compact if and only if every continuous function $f: K \to \mathbb{R}$ is bounded above, that is, there exists $M(f) > 0$ such that $f(x) < M(f)$ for all $x \in X$. (Note: You may assume that the Euclidean distance function is continuous.)
4. Prove that if a sequence of functions \((f_n: X \to Y)\) converges uniformly to \(f: X \to Y\) and \(g: Y \to Z\) is a uniformly continuous function, then \(g f_n: X \to Z\) where \(g f_n(x) = g(f_n(x))\) converges uniformly to \(g f: X \to Z\) where \(g f(x) = g(f(x))\).
5. Let $Y$ be a subset of a space $X$ and let $S$ be a subset of $X$ such that $Y \subseteq S \subseteq cl(Y)$, where $cl(Y)$ is the closure of $Y$. Prove that if $Y$ is connected, then $S$ is also connected.