Concepts and notation

Set theory

If $X$ is a set (class), then $r \in X$ means that $r$ belongs to $X$ or $r$ is a member of $X$.

$\emptyset$ is the empty set, i.e., the set that has no elements.

Two sets (classes) $X$ and $Y$ are called equipotent if there is a one-to-one correspondence between $X$ and $Y$.

If $X$ and $Y$ are sets (classes), then $Y \subseteq X$ means that $Y$ is a subset (subclass) of $X$, i.e., $Y$ is a set such that all elements of $Y$ belong to $X$, and $X$ is a superset of $Y$. A subset is proper if it does coincide with the whole set.

The intersection $Y \cap X$ of two sets (classes) $Y$ and $X$ is the set (class) that consists of all elements that belong both to $Y$ and to $X$.

The union $Y \cup X$ of two sets (classes) $Y$ and $X$ is the set (class) that consists of all elements from $Y$ and from $X$. The union $Y \cup X$ is called disjoint if $Y \cap X = \emptyset$.

The union $\bigcup_{i \in I} X_i$ of sets (classes) $X_i$ is the set (class) that consists of all elements from all sets (classes) $X_i$, $i \in I$.

The intersection $\bigcap_{i \in I} X_i$ of sets (classes) $X_i$ is the set (class) that consists of all elements that belong to each set (class) $X_i$, $i \in I$.

The difference $Y \setminus X$ of two sets (classes) $Y$ and $X$ is the set (class) that consists of all elements that belong to $Y$ but does not belong to $X$.

If a set (class) $X$ is a subset of a set (class) $Y$, i.e., $X \subseteq Y$, then the difference $Y \setminus X$ is called the complement of the set (class) $X$ in the set (class) $Y$ and is denoted by $C_Y X$. 
If $X$ is a set, then $2^X$ is the \textit{power set} of $X$, which consists of all subsets of $X$. The \textit{power set} of $X$ is also denoted by $\mathcal{P}(X)$.

If $X$ and $Y$ are sets (classes), then $X \times Y = \{(x, y); x \in X, y \in Y\}$ is the \textit{Cartesian product} of $X$ and $Y$, in other words, $X \times Y$ is the set (class) of all pairs $(x, y)$, in which $x$ belongs to $X$ and $y$ belongs to $Y$.

A mapping (function) $f$ from a set $X$ into a set $Y$ is denoted by $f: X \rightarrow Y$ and a mapping of an element $x$ into an element $y$ is denoted by $x \mapsto y$.

$Y^X$ denotes the set of all mappings from $X$ into $Y$.

$$X^n = \underbrace{X \times X \times \ldots \times X}_{n}$$

Elements of the set $X^n$ have the form $(x_1, x_2, \ldots, x_n)$ with all $x_i \in X$ and are called $n$-tuples, or simply, tuples.
A fundamental structure of mathematics is a function. However, functions are special kinds of binary relations between two sets, which are defined below.

A binary relation $T$ between sets $X$ and $Y$, also called a correspondence from $X$ to $Y$, is a subset of the Cartesian product $X \times Y$. The set $X$ is called the domain of $T$ ($X = \text{Dom}(T)$) and $Y$ is called the codomain of $T$ ($Y = \text{Codom}(T)$). The range of the relation $T$ is $\text{Rg}(T) = \{ y ; \exists x \in X \ (x, y) \in T \}$). The domain of definition also called the definability domain of the relation $T$ is $\text{DDom}(T) = \{ x ; \exists y \in Y \ ((x, y) \in T) \}$. If $(x, y) \in T$, then one says that the elements $x$ and $y$ are in relation $T$, and one also writes $T(x, y)$.

The image $T(x)$ of an element $x$ from $X$ is the set $\{ y; (x, y) \in T \}$ and the coimage $T^{-1}(y)$ of an element $y$ from $Y$ is the set $\{ x; (x, y) \in T \}$.

The graph of binary relation $T$ between sets of real numbers is the set of points in the two dimensional vector space (a plane), the coordinates of which satisfy this relation.

Binary relations are also called multivalued functions ( mappings or maps).

Taking binary relations $T \subseteq X \times Y$ and $R \subseteq Y \times Z$, it is possible to build a new binary relation $RT \subseteq X \times Z$ that is called the (sequential) composition or superposition of binary relations $T$ and $R$ and is defined as

$$R \circ T = \{ (x, z); x \in X, z \in Z; \text{ where } (x, y) \in T \text{ and } (y, z) \in R \text{ for some } y \in Y \}.$$ 

A preorder (also called quasiorder) on a set (class) $X$ is a binary relation $Q$ on $X$ that satisfies the following axioms:

**O1.** $Q$ is reflexive, i.e. $xQx$ for all $x$ from $X$.

**O2.** $Q$ is transitive, i.e., $xQy$ and $yQz$ imply $xQz$ for all $x, y, z \in X$.

A preorder can be partial or total when for all $x, y \in X$, we have either $xQy$ or $yQx$.

A partial order is a preorder that satisfies the following additional axiom:

**O3.** $Q$ is antisymmetric, i.e., $xQy$ and $yQx$ imply $x = y$ for all $x, y \in X$.

A strict also called sharp partial order is a preorder that is not reflexive, is transitive and satisfies the following additional axiom:

**O4.** $Q$ is asymmetric, i.e., only one relation $xQy$ or $yQx$ is true for all $x, y \in X$.

A linear or total order is a strict partial order that satisfies the following additional axiom:

**O5.** We have either $xQy$ or $yQx$ for all $x, y \in X$. 
A set (class) $X$ is \textit{well-ordered} if there is a partial order on $X$ such that any its non-empty subset has the least element. Such a partial order is called \textit{well-ordering}.

An \textit{equivalence} on a set (class) $X$ is a binary relation $Q$ on $X$ that is reflexive, transitive and satisfies the following additional axiom:

\textbf{O6.} $Q$ is \textit{symmetric}, i.e., $xQy$ implies $yQx$ for all $x$ and $y$ from $X$.

\textbf{Set-theoretical symbols}

\begin{itemize}
  \item $>$ \textit{larger than}
  \item $<$ \textit{less than}
  \item $\geq$ \textit{larger than or equal to}
  \item $\leq$ \textit{less than or equal to}
  \item $=$ \textit{equal}
  \item $\approx$ \textit{approximately equal}
  \item $\neq$ \textit{not equal}
  \item $\in$ \textit{belongs}
  \item $\notin$ \textit{does not belong}
  \item $\subseteq$ \textit{is a subset}
  \item $\subset$ \textit{is a proper subset}
  \item $\subsetneq$ \textit{is not a proper subset}
\end{itemize}
A function (also called a mapping or map or total function or total mapping or everywhere defined function) \( f \) from \( X \) to \( Y \) is defined as a binary relation between sets \( X \) and \( Y \) that satisfies two conditions: (1) there are no elements from \( X \) which are corresponded to more than one element from \( Y \), and (2) to any element from \( X \), some element from \( Y \) is corresponded. Traditionally, the function \( f \) is also denoted by \( f: X \rightarrow Y \) or by \( f(x) \). In the latter formula, \( x \) is a variable that takes values in \( X \). The support, or carrier, of a numerical function \( f \) is the closure of the set where \( f(x) \neq 0 \). Usually the element \( f(a) \) is called the image of the element \( a \) and denotes the value of \( f \) on the element \( a \) from \( X \). The coimage \( f^{-1}(y) \) of an element \( y \) from \( Y \) is the set \( \{ x; f(x) = y \} \).

However, the traditional definition does not include all kinds of functions and their representations.

There are three basic forms of function representation (definition):

1. (The set-theoretical, e.g., table, representation) A function \( f \) is given as a subset \( R_f \) of the direct product \( X \times Y \) such that the first element if each pair from \( R_f \) uniquely defines the second element in this pair, e.g., in a form of a table or of a list of pairs \( (x, y) \) where the first element \( x \) is taken from \( X \), while the second element \( y \) is the image \( f(x) \) of the first one. The set \( R_f \) is called the graph of the function \( f \). When \( X \) and \( Y \) are sets of points in a geometrical space, e.g., their elements are real numbers, the graph of the function \( f \) is called the geometrical graph of \( f \).

2. (The analytic representation) A function \( f \) is described by a formula, i.e., a relevant expression in a mathematical language, e.g., \( f(x) = \sin(e^x + \cos x) \).

3. (The algorithmic representation) A function \( f \) is given as an algorithm that computes \( f(x) \) given \( x \).

\[ f(x) \equiv a \] means that the function \( f(x) \) is equal to \( a \) at all points where \( f(x) \) is defined.

A function (mapping) \( f \) from \( X \) to \( Y \) is an injection if the equality \( f(x) = f(y) \) implies the equality \( x = y \) for any elements \( x \) and \( y \) from \( X \), i.e., different elements from \( X \) are mapped into different elements from \( Y \).

A function (mapping) \( f \) from \( X \) to \( Y \) is a projection also called surjection if for any \( y \) from \( Y \) there is \( x \) from \( X \) such that \( f(x) = y \).

A function (mapping) \( f \) from \( X \) to \( Y \) is a bijection if it is both a projection and injection.
A function (mapping) \( f \) from \( X \) to \( Y \) is an *inclusion* if the equality \( f(x) = x \) holds for any element \( x \) from \( X \).

Two important concepts of mathematics are the *domain* and *range* of a function. If \( f \) is a function from \( X \) into \( Y \), then the set \( X \) is called the *domain* of \( f \) (it is denoted by \( \text{Dom} f \)) and \( Y \) is called the *codomain* of \( f \) (it is denoted by \( \text{Codom} f \)). The *range* \( \text{Rg} f \) of the function \( f \) is the set of all elements from \( Y \) assigned by \( f \) to, at least, one element from \( X \), or formally, \( \text{Rg} f = \{ y; \exists x \in X \ (f(x) = y) \} \). The *domain of definition* also called the *definability domain*, \( \text{DDom} f \), of the function \( f \) is the set of all elements from \( X \) that related by \( f \) to, at least, one element from \( Y \) is or formally, \( \text{DDom} f = \{ x; \exists y \in Y \ (f(x) = y) \} \). Thus, for a partial function \( f \), its domain of definition \( \text{DDom} f \) is the set of all elements for which \( f(x) \) is defined.

Taking two mappings (functions) \( f \colon X \to Y \) and \( g \colon Y \to Z \), it is possible to build a new mapping (function) \( g \circ f \colon X \to Z \) that is called the *(sequential) composition* or *superposition* of mappings (functions) \( f \) and \( g \) and defined by the rule \( g(f(x)) \) for all \( x \) from \( X \).

For any set \( S \), \( \chi_S(x) \) is the *characteristic function*, also called the *set indicator function*, if \( \chi_S(x) \) is equal to 1 when \( x \in S \) and is equal to 0 when \( x \notin S \), and \( C_S(x) \) is its partial characteristic function if \( C_S(x) \) is equal to 1 when \( x \in S \) and is undefined when \( x \notin S \).

If \( f : X \to Y \) is a function and \( Z \subseteq X \), then the restriction \( f|_Z \) of \( f \) on \( Z \) is the function defined only for elements from \( Z \) and \( f|_Z(z) = f(z) \) for all elements \( z \) from \( Z \).

**Basic functions** used in the course (study their properties)

\[
d^x
\]

\[
\log_a x
\]

\[
kx^n
\]

\[
p(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0
\]
Named sets as the most encompassing and fundamental mathematical construction include ordinary sets and all their generalizations, such as fuzzy sets and multisets, providing unified foundations for the whole mathematics. Functions, mappings, operations, relations, graphs, multigraphs, operators, fiber bundles, morphisms, functors, enumerations and many other mathematical structures are named sets. Moreover, all mathematical structures are built of named sets.

A named set (also called a fundamental triad) has the following graphic representation

Entity 1 \quad \text{connection} \quad \rightarrow \quad \text{Entity 2} \quad (1)

or

Essence 1 \quad \text{correspondence} \quad \rightarrow \quad \text{Essence 2} \quad (2)

In the fundamental triad (named set) (1) or (2), Entity 1 (Essence 1) is called the support, the Entity 2 (Essence 2) is called the reflector (also called the set or component of names) and the connection (correspondence) between Entity 1 (Essence 1) and connection (correspondence) is called the reflection (also called the naming correspondence) of the fundamental triad (1) (respectively, (2)).

In the symbolic form, a named set (also called a fundamental triad) $X$ is a triad $(X, f, I)$ where $X$ is the support of $X$ and is denoted by $S(X)$, $I$ is the component of names (also called set of names or reflector) of $X$ and is denoted by $N(X)$, and $f$ is the naming correspondence (also called reflection) of the named set $X$ and is denoted by $n(X)$. The most popular type of named sets is a named set $X = (X, f, I)$ in which $X$ and $I$ are sets and $f$ consists of connections between their elements. When these connections are set theoretical, i.e., each connection is represented by a pair $(x, a)$ where $x$ is an element from $X$ and $a$ is its name from $I$, we have a set theoretical named set, which is binary relation.

Two model examples of a named set:

$X$ is a group of people, $N$ is a set of their names and $f$ is the connection between people and their names.
$X$ is a collection of Internet resources, $N$ is a set of their Internet names and $f$ is the connection between resources and their names.

**Logical concepts and structures**

If $P$ and $Q$ are two statements, then $P \rightarrow Q$ means that $P$ implies $Q$ and $P \leftrightarrow Q$ means that $P$ is equivalent to $Q$.

**Logical operations:**

- *negation* is denoted by $\neg$ or by $\sim$,
- *conjunction* also called logical “and” is denoted by $\land$ or by $\&$ or by $\cdot$,
- *disjunction* also called logical “or” is denoted by $\lor$,
- *implication* is denoted by $\rightarrow$ or by $\Rightarrow$ or by $\supset$,
- *equivalence* is denoted by $\leftrightarrow$ or by $\equiv$ or by $\Leftrightarrow$.

The logical symbol $\forall$ is called the *universal quantifier* and means “for any”.

The logical symbol $\exists$ is called the *existential quantifier* and means “there exists”.